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Dual attachment pairs in categorically-algebraic topology

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Abstract

The paper is a continuation of our study on developing a new approach to (lattice-valued) topological structures, which relies on category theory and universal algebra, and which is called categorically-algebraic (catalg) topology. *The new framework is used to build a topological setting, based in a catalg extension of the set-theoretic membership relation "∈" called* dual attachment, *thereby dualizing the notion of* attachment *introduced by the authors earlier. Following the recent interest of the fuzzy community in* topological systems *of S. Vickers, we clarify completely relationships between these structures and (dual) attachment, showing that unlike the former, the latter have no inherent topology, but are capable of providing a natural transformation between two topological theories. We also outline a more general setting for developing the attachment theory, motivated by the concept of* (L, M)-fuzzy topological space *of T. Kubiak and A. Šostak.*

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1. Introduction

Motivated by the abundance of lattice-valued topological theories available in the literature and the lack of interaction means between them, this paper makes another step towards developing a new approach to (lattice-valued) topological structures deemed to incorporate in itself the majority of the existing settings. Based in category theory and universal algebra, the new framework is called *categorically-algebraic* (*catalg*) *topology* [64] to underline its generating theories. It originates from *point-set lattice-theoretic* (*poslat*) *topology* of S. E. Rodabaugh [61, 62], developed in the framework of lattice-valued powerset theories (motivated by algebraic theories (in clone form) of E. G. Manes [46], the basic example given by the theory of the powerset functor on the category **Set** of sets and maps, appended with its induced contravariant powerset functor), where the underlying algebraic structures for topology are semi-quantales. We replace semi-quantales with algebras (possibly having a class of non-finitary operations) from an arbitrary variety and consider an abstract category as the ground for topology. The framework obtained in this manner includes the most important approaches to (lattice-valued) topology, providing convenient means of intercommunication between them, and (that is more essential) ultimately erasing the border between lattice-valued and crisp developments. Moreover, the amount of building blocks of the proposed theory is reduced to minimum, postulating the so-called "plug and play approach", when additional requirements on the underlying setting are motivated by the need of additional properties. In particular, we never employ the framework of *monadic topology*, developed by several authors in the literature [18, 29, 31], as being too restrictive for our current purposes. Briefly speaking, we propagate the slogan: *achieve more with less*. On the other hand, it should be underlined immediately, that all essential properties of modern (lattice-valued) topology (e.g., compactness, separation axioms, connectedness, *etc.*) can be incorporated in the catalg setting. It is the theory of *catalg spaces* [67], which is currently undertaking the job. Moreover, the new framework is rapidly progressing in several other directions [65, 71, 72, 73, 74], influencing each other dramatically. It is the main purpose of this paper to show one of the important applications of the new theory, i.e., the development of a fruitful topological setting, based in a catalg generalization of the set-theoretic membership relation "$\in$".

The starting point for the proposed research topic lies in the concept of *quasi-coincidence* between a fuzzy point and a fuzzy set, introduced by P.-M. Pu and Y.-M. Liu [49, Definition 2.3′] with the aim to extend the standard approach to topology through neighborhood structures by fuzzifying the above relation "$\in$". Given a set X, a *fuzzy point* a_x (a map from X to the unit interval $\mathbb{I} = [0, 1]$, taking value a at x and 0 elsewhere) is said to be *quasi-coincident* with a *fuzzy set* α (a map $X \xrightarrow{\alpha} \mathbb{I}$) provided that $1 - \alpha(x) < a$. Later on, Y.-M. Liu and M.-K. Luo [43, Definition 2.3.1] used a completely distributive lattice L, equipped with an order-reversing involution $(-)'$, to generalize the definition to $a \not\leqslant (\alpha(x))'$. Moreover, Y.-M. Liu [42] showed that the quasi-coincidence relation is the unique membership relation, which satisfies the four principles of a “reasonable” (generating a fruitful topological theory) membership relation.

The next step was done by C. Guido (and V. Scarciglia) [25, 26, 27], who removed the requirement on the existence of an involution and introduced a lattice-valued analogue of the relation in question under the name of *attachment*. Given a complete lattice L, an *attachment* $\mathcal{A}$ in L is a family $\{F_a \,|\, a \in L\backslash\{\bot\}\}$ of *completely prime* ($\bigvee S \in F_a$ implies $S \bigcap F_a \neq \varnothing$) filters of L, indexed by its elements, with an additional requirement $F_{\bot} = \varnothing$. An L-point a_x is said to be *attached* to an L-set α (denoted $a_x \,\mathcal{A}\, \alpha$) provided that $\alpha(x) \in F_a$. The new notion not only generalizes quasi-coincidence relation (take $L = \mathbb{I}$ and let $F_a = \{b \,|\, 1 - a < b\}$ for every $a \in L$), but also induces a functor from the category L-**Top** of L-topological spaces [33] to the category **Top** of topological spaces, which takes an L-topological space (X, τ) to the crisp space $(S_X, \tau^{\star})$, where S_X is the set of L-points of X and $\tau^{\star} = \{\alpha^{\star} \,|\, \alpha \in \tau\}$ with $\alpha^{\star} = \{a_x \,|\, a_x \,\mathcal{A}\, \alpha\}$. The new functor is closely related to the well-known *hypergraph functors* [20, 32, 36, 53, 55, 58] used in lattice-valued mathematics, bringing them under the common roof of attachment, and thereby removing the difference in their definition by various authors, that prevents them from gaining in popularity in applications. The employed machinery is based in the concept of *topological system* of S. Vickers [75], introduced to merge point-set (topological spaces) and pointless (their underlying algebraic structures - *locales* [35]) topology, which recently has raised an interest among the fuzzy researchers [12, 13, 14, 74] as a possible framework to incorporate both lattice-valued topology and its underlying algebraic structures (in most cases, particular kinds of the already mentioned semi-quantales).

In [73], S. Solovyov generalized the approach even further, taking into consideration the fact that there exists a one-to-one correspondence between completely prime filters of a complete lattice L and *points* of L (frame homomorphisms from L to the two-element frame $\mathbf{2} = \{\bot, \top\}$ [35]), which opens a possibility to define an attachment as a map $L \xrightarrow{\mathcal{A}} \mathbf{Frm}(L, \mathbf{2})$ (omitting the requirement $F_{\bot} = \varnothing$ as never influencing the essential properties of the theory). The above-mentioned catalg approach to topology in hand, he introduced the notion of variable-basis attachment for an arbitrary variety of algebras.

Definition 1.1. Let $\mathbf{A}$ be a variety of algebras and let $\mathbf{A} \xrightarrow{(-)^{*}} \mathbf{Set}^{op}$ be a functor, which takes an $\mathbf{A}$-algebra A to its underlying set. An ($\mathbf{A}$-)*attachment* is a triple $F = (\Omega\, F, \Sigma\, F, \Vdash)$, where $\Omega\, F$ and $\Sigma\, F$ are $\mathbf{A}$-algebras, and $\Omega\, F \xrightarrow{\Vdash} \mathbf{A}(\Omega\, F, \Sigma\, F)$ is a map. An *attachment morphism* $F_1 \xrightarrow{f} F_2$ is a pair of $\mathbf{A}$-homomorphisms $(\Omega\, F_1, \Sigma F_1) \xrightarrow{(\Omega\, f, \Sigma\, f)} (\Omega\, F_2, \Sigma\, F_2)$ such that for every $a_1 \in \Omega\, F_1$ and every $a_2 \in \Omega\, F_2$, $(\Vdash_2(a_2))(\Omega\, f(a_1)) = (\Sigma\, f \circ \Vdash_1((\Omega\, f)^{*op}(a_2)))(a_1)$. $\mathbf{AttA}$ is the category of attachments and their homomorphisms, concrete over the product category $\mathbf{A} \times \mathbf{A}$.

The main achievement of [73] is a common framework (based in catalg attachment) for the majority of instances of hypergraph functor, providing a convenient tool for exploring their features, and the explicit study of categorical properties of attachment and its generated functors (in the sense of [25, 26]), which appear to have a right adjoint for a particular attachment type called *spatial*, generalizing the respective property of the hypergraph functor of U. Höhle [32]. The advantage of the last result is the extension of the achievement of U. Höhle to all of the above instances of hypergraph functor, taking the appropriate underlying variety in each case. Moreover, this fact illustrates the main contribution of the catalg setting itself, whose essence is: *prove once for many*. It is the goal of this paper to continue the once started line of research by presenting a dual version of attachment. The meta-mathematical inducement for the new approach was given by the observation that both partially ordered sets and categories provide the means for dualization of their results. The real push, however, was taken up by the authors in their wish to change the setting of lattice-valued attachment of [26] from filters to ideals. After a brief discussion on the topic, the following crucial observations came to light.

Observation 1. The case of a complete chain L provides a possibility to define an attachment $\mathcal{A}$ on L as a family $F_\perp = \varnothing$ and $F_a = \{b \in L \mid a < b\}$ for every $a \in L \backslash \{\perp\}$. More particularly, given an L-point a_x and an L-set α, $a_x \, \mathcal{A} \, \alpha$ iff $\alpha(x) \in F_a$ iff $a < \alpha(x)$. If $\alpha(x) \neq \top$, then $G_{\alpha(x)} = \{b \in L \mid b < \alpha(x)\}$ is a completely prime ideal of L, which is the notion dual to that of completely prime filter. This suggests the family $G_\top = \varnothing$ and $G_a = \{b \in L \mid b < a\}$ for every $a \in L \backslash \{\top\}$ as a possible substitute for $\mathcal{A}$, thereby turning the attachment condition $a_x \ \mathcal{A} \ \alpha$ from $\alpha(x) \in F_a$ into $a \in G_{\alpha(x)}$.

Observation 2. In the wake of [73], the concluding section of [26] introduced the notion of generalized attachment, based in the category **QFrm** of *quasi-frames* (Definition 2.2 of this paper), which are complete lattices, with the respective morphisms preserving arbitrary $\bigvee$ and binary $\wedge$ (the empty meet is excluded). The new category gives rise to the notion of completely prime *quasi-filter* of a complete lattice L as the preimage of $\{\top\}$ under a quasi-frame map (*quasi-point*) $L \xrightarrow{p} \mathbf{2}$, which allows, apart from the standard filters, also the empty one. A *generalized attachment* in a quasi-frame L is then a map $L \xrightarrow{\Vdash} \mathbf{QFrm}(L, \mathbf{2})$ (cf. Definition 1.1). The particular case of a complete chain L suggests the following definition of the map $\Vdash$:

$$(\Vdash(a))(b) = \begin{cases} \top, & a < b, \\ \perp, & \text{otherwise}, \end{cases}$$

providing a generalized attachment $F = (L, \mathbf{2}, \Vdash)$. Brief consideration brings a new map $L \xrightarrow{\Vvdash} \mathbf{2}^L$ defined by $(\Vvdash(a))(b) = (\Vdash(b))(a)$, which induces the triple $G = (L, \mathbf{2}, \Vvdash)$. An important property of the map is its preservation of $\bigvee$ and binary $\wedge$, i.e., $(\Vvdash(\bigvee S))(b) = \top$ iff $b < \bigvee S$ iff $b < s$ for some $s \in S$ iff $(\Vvdash(s))(b) = \top$ for some $s \in S$ iff $(\bigvee_{s \in S} \Vvdash(s))(b) = \top$, whereas $(\Vvdash(s \wedge t))(b) = \top$ iff $b < s \wedge t$ iff $b < s$, $b < t$ iff $(\Vvdash(s))(b) \wedge (\Vvdash(t))(b) = \top$. On the other hand, the map $\Vvdash(a)$ is not $\bigvee$-preserving for $a \neq \perp$, since $(\Vvdash(a))(\perp) = \top \neq \perp$. In such a manner, the so-called *dual attachment pair* (F, G) arises, where duality means $a_x \, F \, \alpha$ iff $(\Vdash(a))(\alpha(x)) = \top$ iff $\Vvdash(\alpha(x))(a) = \top$ iff $a_x \, G \, \alpha$.

The above remarks provide an opening for a new definition of attachment, called in this paper *dual attachment*. Apart from concrete applications to the already developed theory, the concept represents a catalg extension of the notion of "duality" in mathematics (should not be mixed with the theory of *catalg dualities* [71, 72] dealing with topological representations of algebraic structures). The attentive reader will see that catalg "duality" is neither categorical duality (as, e.g., the dual of a category), nor algebraic duality (as, e.g., the dual of a partially ordered set), but truly categorically-algebraic "duality". It will be the topic of our forthcoming papers to find the proper place for such kind of dualities in mathematics, whereas this manuscript is bound to consider categorical properties of dual attachment and the functors arising from it. It appears that the concept still retains a close relation to topological systems of S. Vickers. On the other hand, the results of this paper clearly show that the nature of the two notions is essentially different, the latter being equipped with an internal topology extracted by the procedure of *spatialization of systems* introduced by S. Vickers [75], whereas the former providing a way of interaction (natural transformation) between two topological theories, resulting in a functor between the categories of the respective topological structures. The achievement finally resolves the question (posed in the fuzzy community) on relationships between the two concepts and a possible common framework for both of them (non-existent due to the principally different categorical perspectives of the notions). Moreover, the just mentioned crucial property of attachment gave rise to the study of one of the authors on general relationships between catalg topological theories and their induced catalg topological structures (see Section 6 of this manuscript for the respective definitions), partly announced during the presentation of [65] and currently being developed as the subject of a forthcoming paper, similar by the approach (but not the results) to the widely used in categorical algebra *algebraic theories* of F. W. Lawvere [41].

This paper uses both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [1, 28, 45, 46]. For the notions of universal algebra we recommend [7, 9, 23, 46]. Although the authors tried to make the paper as much self-contained as possible, some details are still omitted and left to the reader.

2. Dual attachment with its induced categories and functors

In this section, we introduce the notion of *dual attachment* and consider its related categories and functors. The cornerstone of the approach is the concept of *algebra*. The structure is to be thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also (that is different from the standard theory of universal algebra) complete lattice, frame, quantale. The classes of *finitary algebras* (those induced by a set of finitary operations) are usually described in universal algebra as either *varieties* or *equational classes* [7, 9, 23], which coincide due to the well-known

HSP-theorem of G. Birkhoff [6]. To incorporate the algebraic structures used in lattice-valued topology (where set-theoretic unions are usually replaced by arbitrary joins), this paper extends the approach of varieties to cover its needs.

Definition 2.1.

(1) Let $\boldsymbol{\Omega} = (n_\lambda)_{\lambda\in\Lambda}$ be a (possibly proper) class of cardinal numbers. An *$\boldsymbol{\Omega}$-algebra* is a pair $(A, (\omega^A_\lambda)_{\lambda\in\Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega^A_\lambda} A$ (*n_λ-ary primitive operations* on A). An *$\boldsymbol{\Omega}$-homomorphism* $(A, (\omega^A_\lambda)_{\lambda\in\Lambda}) \xrightarrow{\varphi} (B, (\omega^B_\lambda)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi \circ \omega^A_\lambda = \omega^B_\lambda \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$. $\mathbf{Alg}(\boldsymbol{\Omega})$ is the construct of $\boldsymbol{\Omega}$-algebras and $\boldsymbol{\Omega}$-homomorphisms.

(2) Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of $\boldsymbol{\Omega}$-homomorphisms with injective (resp. surjective) underlying maps. A *variety of $\boldsymbol{\Omega}$-algebras* is a full subcategory of $\mathbf{Alg}(\boldsymbol{\Omega})$ closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).

(3) Given a variety $\mathbf{A}$, a *reduct* of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\boldsymbol{\Omega}_{\mathbf{B}} \subseteq \boldsymbol{\Omega}_{\mathbf{A}}$ and $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

From now on, every concrete category is supposed to be equipped with the underlying functor $|-|$ to its respective ground category (cf. Definition 1.1). For the sake of shortness, the fact will be never mentioned explicitly again.

An experienced reader will probably be able to find numerous examples to back the new notion. Below, we extend the list with several more items, all of which (except the last one) come from the realm of lattice-valued topology [61, 62] and will be used throughout the paper.

Definition 2.2.

(1) Given $\Xi \in \{\bigvee, \bigwedge\}$, a *$\Xi$-semilattice* is a partially ordered set having arbitrary Ξ. $\mathbf{CSLat}(\Xi)$ is the variety of Ξ-semilattices.

(2) A *semi-quantale* (*s-quantale*) is a $\bigvee$-semilattice equipped with a binary operation $\otimes$ (*multiplication*). **SQuant** is the variety of s-quantales.

(3) An s-quantale is called *DeMorgan* provided that it is equipped with an order-reversing involution $(-)'$. **DmSQuant** is the variety of DeMorgan s-quantales.

(4) An s-quantale is called *unital* (*us-quantale*) provided that its multiplication has the unit 1. **USQuant** is the variety of us-quantales.

(5) A *quantale* is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides. **Quant** is the variety of quantales.

(6) A *quasi-frame* (*q-frame*) is an s-quantale whose multiplication is $\wedge$. **QFrm** is the variety of q-frames.

(7) A *semi-frame* (*s-frame*) is a unital q-frame. **SFrm** is the variety of s-frames.

(8) A *frame* is an s-frame which is a quantale. **Frm** is the variety of frames.

(9) A *closure semilattice* (*c-semilattice*) is a $\bigwedge$-semilattice, with the singled out bottom element $\bot$. **CSL** is the variety of c-semilattices.

The reader should bear in mind that all varieties of Definition 2.2 have complete lattices as objects. Moreover, all of them except **DmSQuant**, **Quant** and **Frm** are reducts of the variety **CLat** of complete lattices. To continue the topic, we remark that **CSLat**$(\bigvee)$ is a reduct of **SQuant**; **SQuant** is a reduct of **USQuant** and **QFrm**; **USQuant** is a reduct of **SFrm**; **Set** is a reduct of any variety. Also notice that the categories **SFrm** and **QFrm**, having essentially the same objects (complete lattices), differ significantly on morphisms. The last item of Definition 2.2 was motivated by the concept of *strong* ($\top$-preserving) quantale homomorphism [37, 38], and would provide an additional example for the concept of catalg topology introduced later on in the paper.

For the sake of convenience, from now on we use the following notations, which differ from the respective category-theoretic ones (see, e.g., [14, 57, 61] for the motivation). An arbitrary variety is denoted **A**, **B**, **C**, *etc.* The categorical dual of a variety **A** is denoted **LoA** (the "**Lo**" comes from "localic"), whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Several other categories introduced in the paper (but always related to varieties) employ similar notation for their duals. Following the already accepted designation of [35], the dual of **Frm** is denoted **Loc** , whose objects are called *locales*. To distinguish maps (or, more generally, morphisms) and homomorphisms, the former are denoted f, g, h, reserving φ, ψ, ϕ for the latter. Given a homomorphism φ, the respective localic one is denoted φ^{op} and vice versa. Given an algebra A of a variety **A** (or an object of a related category), $\mathbf{S}_A$ stands for the subcategory of **LoA** comprising the identity 1_A on A as the only morphism. We will occasionally use the notation $\mathbf{S}_A^{\mathbf{A}}$, to underline the originating variety of the algebra A. Given a set X, an algebra A and an element $a \in A$, $X \xrightarrow{a_X} A$ denotes the constant map with value a.

A few words are due to the many-valued framework employed in the paper. Following [73], we extend the concept of lattice-valued set to that of *algebraic* one, which is defined as follows (recall the underlying functor of $\mathbf{Alg}(\boldsymbol{\Omega})$).

Definition 2.3. Let X be a set and let A be an algebra of a variety **A**. An (A-)*algebraic set* in X is a map $X \xrightarrow{\alpha} |A|$.

The underlying idea of the new setting is based in a direct *algebraization* of the classical frameworks of L. A. Zadeh [77] and J. A. Goguen [21], which can be easily restored by choosing an appropriate variety **A**. Despite the fact that the theory of algebraic sets provides a nice challenge for research, the current paper will not develop the topic off the bounds of its interests. To distinguish algebraic sets from other maps, from now on, they will be denoted α, β, γ.

All preliminaries in their places, the new notion of attachment is ready to introduce (recall that **Set** stands for the category of sets and maps).

Definition 2.4. Let **B** be a variety and let $\mathbf{B} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ be a functor such that $B^* = |B|$. A *dual* (**B**-)*attachment* is a triple $G = (\Omega\, G, \Sigma\, G, \Vdash)$, where

ΩG, ΣG are **B**-algebras, and $\Omega G \xrightarrow{\Vdash} \Sigma G^{|\Omega G|}$ is a **B**-homomorphism. A *dual attachment morphism* $G_1 \xrightarrow{f} G_2$ is then a pair of **B**-homomorphisms $(\Omega G_1, \Sigma G_1) \xrightarrow{(\Omega f, \Sigma f)} (\Omega G_2, \Sigma G_2)$ such that for every $b_1 \in \Omega G_1$ and every $b_2 \in \Omega G_2$, $(\Vdash_2(\Omega f(b_1)))(b_2) = (\Sigma f \circ \Vdash_1(b_1))((\Omega f)^{*op}(b_2))$. **ATTB** is the category of dual attachments and their homomorphisms, concrete over the product category $\mathbf{B} \times \mathbf{B}$.

To convince the reader that Definition 2.4 gives a category, we check the closure under composition. Given two **ATTB**-morphisms $G_1 \xrightarrow{f} G_2$, $G_2 \xrightarrow{g} G_3$ and $b_1 \in \Omega G_1$, $b_3 \in \Omega G_3$, one easily gets that $(\Vdash_3(\Omega(g \circ f)(b_1)))(b_3) = \Sigma g \circ (\Vdash_2(\Omega f(b_1)))((\Omega g)^{*op}(b_3)) = \Sigma g \circ \Sigma f \circ (\Vdash_1(b_1))((\Omega f)^{*op} \circ (\Omega g)^{*op}(b_3)) = \Sigma(g \circ f) \circ (\Vdash_1(b_1))((\Omega g \circ \Omega f)^{*op}(b_3)) = (\Sigma(g \circ f) \circ (\Vdash_1(b_1)))(((\Omega(g \circ f))^{*op})(b_3))$.

An attentive reader will notice striking similarities between the categories **AttA** (Definition 1.1) and **ATTB** (to distinguish the new type of attachment, capital letters are used in the notation of the respective category). A somewhat deeper insight into their nature reveals not less striking differences in their behavior, one of which being ready for display on the spot. In [73], a full embedding $\mathbf{A} \xhookrightarrow{\mathrm{E}_{\mathbf{A}}} \mathbf{AttA}$ was provided, showing that **AttA** gave a proper extension of its underlying variety **A**. In the new framework, a similar procedure results in an (in general, non-full) embedding under certain requirements only.

Proposition 2.5. *Suppose there exists a nullary operation ω_{λ_0} of **B**, satisfying the identity $\omega_\lambda(\langle \omega_{\lambda_0} \rangle_{n_\lambda}) = \omega_{\lambda_0}$ for every **B**-operation ω_λ (implying that ω_{λ_0} is the unique nullary operation of **B**). Then there exists an (in general, non-full) embedding $\mathbf{B} \xhookrightarrow{\mathrm{E}_{\mathbf{B}}} \mathbf{ATTB}$, $\mathrm{E}_{\mathbf{B}}(B_1 \xrightarrow{\varphi} B_2) = (B_1, B_1, \Vdash_1) \xrightarrow{(\varphi,\varphi)} (B_2, B_2 \Vdash_2)$, with $B_i \xrightarrow{\Vdash_i} B_i^{|B_i|}$ given by $\Vdash_i(b) = \underline{\omega_{\lambda_0}^{B_i}}$.*

Proof. To show that the functor is correct on objects, notice that given $\lambda \in \Lambda_{\mathbf{B}}$ and $b_i \in B$ for $i \in n_\lambda$, $(\Vdash(\omega_\lambda^B(\langle b_i \rangle_{n_\lambda})))(b) = \underline{\omega_{\lambda_0}^B}(b) = \omega_{\lambda_0}^B = \omega_\lambda^B(\langle \omega_{\lambda_0}^B \rangle_{n_\lambda}) = \omega_\lambda^B(\langle (\Vdash(b_i))(b) \rangle_{n_\lambda}) = (\omega_\lambda^{B^{|B|}}(\langle \Vdash(b_i) \rangle_{n_\lambda}))(b)$ for every $b \in B$. To check the correctness on morphisms, notice that given $b_1 \in B_1$ and $b_2 \in B_2$, one gets, $(\Vdash_2(\varphi(b_1)))(b_2) = \underline{\omega_{\lambda_0}^{B_2}}(b_2) = \omega_{\lambda_0}^{B_2} = \varphi(\omega_{\lambda_0}^{B_1}) = (\varphi \circ \underline{\omega_{\lambda_0}^{B_1}})(\varphi^{*op}(b_2)) = (\varphi \circ (\Vdash_1(b_1)))(\varphi^{*op}(b_2))$. The embedding properties of $\mathrm{E}_{\mathbf{B}}$ follow directly from the definition of the functor. The claim on non-fullness requires an additional assumption that there exist two different **B**-homomorphisms $B_1 \overset{\varphi}{\underset{\psi}{\rightrightarrows}} B_2$. Then $\mathrm{E}_{\mathbf{B}}(B_1) \xrightarrow{(\varphi,\psi)} \mathrm{E}_{\mathbf{B}}(B_2)$ is an **ATTB**-morphism, since given $b_1 \in B_1$ and $b_2 \in B_2$, $(\Vdash_2(\varphi(b_1)))(b_2) = \underline{\omega_{\lambda_0}^{B_2}}(b_2) = \omega_{\lambda_0}^{B_2} = \psi(\omega_{\lambda_0}^{B_1}) = (\psi \circ \underline{\omega_{\lambda_0}^{B_1}})(\varphi^{*op}(b_2)) = (\psi \circ (\Vdash_1(b_1)))(\varphi^{*op}(b_2))$. By the fact that $\varphi \neq \psi$, we obtain that (φ, ψ) is not in the image of $\mathrm{E}_{\mathbf{B}}$, thereby concluding the proof of the proposition. □

An example for Proposition 2.5 is the variety $\mathbf{CSLat}(\bigvee)$ of $\bigvee$-semilattices, which gives rise to the respective non-full embedding. The variety **Frm** of frames, however, does not fit into the proposed framework, having more than one nullary operation, but its reduct **QFrm** suits well. In one word, in some cases the category **ATTB** provides a proper extension of its underlying variety.

To continue, we need additional notions from the framework of *categorically-algebraic* (*catalg*) *topology*, introduced recently [64] as an extension of the *point-set lattice-theoretic* (*poslat*) *topology* of S. E. Rodabaugh [57, 61]. The full development of the theory will be given in Section 6 of this paper, whereas here, we just borrow some of its building blocks.

By analogy with its predecessor, the new setting is based in a generalization of the backward powerset theory employed by the classical topological setting. The intuition for the new concept comes from the so-called (*pre*)*image* operators [61], well-known for every working mathematician. Recall that given a set map $X \xrightarrow{f} Y$, there exist the maps $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ (resp. $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$) such that $f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}$ (resp. $f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}$). The operators have already been extended to powersets of lattice-valued sets (see [10, 21, 56, 77]) and the latter one can be lifted to a more general setting.

Proposition 2.6. *Given a variety* **A**, *every subcategory* **C** *of* **LoA** *induces a functor* **Set** $\times$ **C** $\xrightarrow{(-)^{\leftarrow}}$ **LoA** *defined by* $((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$ *with* $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Proof. The proof consists of easy calculations and can be found in [69, 70]. □

For the sake of convenience, the functor **Set** $\times$ $\mathbf{S}_A \xrightarrow{(-)^{\leftarrow}}$ **LoA** (the so-called *fixed-basis approach*, whereas the full framework is referred to as the *variable-basis* approach) is denoted by $(-)^{\leftarrow}_A$, omitting the notation for 1_A in its definition. The functor of Proposition 2.6 has the merit of incorporating in itself the majority of the approaches to powersets of many-valued mathematics. The most crucial of its properties is the fact that it gives rise to a category of catalg (strictly speaking, its *variety-based* reduction [67]) topological spaces, providing a common framework for many approaches to (lattice-valued) topology.

Definition 2.7. Let **A** be a variety and let **C** be a subcategory of **LoA**. A **C**-*topological space* (**C**-*space*) is a triple (X, A, τ), where (X, A) is a **Set** $\times$ **C**-object, and τ (**C**-*topology* on (X, A)) is a subalgebra of A^X. A **C**-*continuous map* $(X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)$ is a **Set** $\times$ **C**-morphism $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$ such that $((f,\varphi)^{\leftarrow})^{\rightarrow}(\tau_2) \subseteq \tau_1$. **C-Top** is the category of **C**-spaces and **C**-continuous maps, which is concrete over the product category **Set** $\times$ **C**. The category $\mathbf{S}_A$-**Top** is denoted A-**Top**, whose objects (resp. morphisms) are shortened to (X, τ) (resp. f).

It should be underlined that the category **C-Top** is a particular instance of a more general approach to catalg topology, developed in Section 6 of this paper. The main advantages of the new framework have already been described in an abstract way in Introduction and would be illustrated by concrete examples in Section 6. At the moment, the reader should notice that apart from serving as a convenient tool for developing the attachment theory, the new setting provides the (much needed) means of interaction between hugely diversified (lattice-valued) topological theories available in the modern literature.

It appears that the framework of attachment provides a more general category for topology than **C-Top**.

Definition 2.8. Given a variety $\mathbf{B}$ and a subcategory $\mathbf{D}$ of $\mathbf{LoATTB}$, a $\mathbf{D}$-*topological space* ($\mathbf{D}$-*space*) is a triple (X, G, τ), where (X, G) is a $\mathbf{Set} \times \mathbf{D}$-object, and τ ($\mathbf{D}$-*topology* on (X, G)) is a subalgebra of $(\Omega\, G)^X$. A $\mathbf{D}$-*continuous map* $(X_1, G_1, \tau_1) \xrightarrow{(f,g)} (X_2, G_2, \tau_2)$ is then a $\mathbf{Set} \times \mathbf{D}$-morphism $(X_1, G_1) \xrightarrow{(f,g)} (X_2, G_2)$ with $((f, (\Omega\, g)^{op})^{\leftarrow})^{\rightarrow}(\tau_2) \subseteq \tau_1$. $\mathbf{D}$-$\mathbf{Top}$ is the category of $\mathbf{D}$-spaces and $\mathbf{D}$-continuous maps, concrete over the category $\mathbf{Set} \times \mathbf{D}$.

For the sake of brevity, the category $\mathbf{S}_G$-$\mathbf{Top}$ is denoted G-$\mathbf{Top}$, employing the shortened notations of the category A-$\mathbf{Top}$. Under the assumption used at the beginning of Proposition 2.5, there exists the embedding functor $\mathbf{LoB\text{-}Top} \overset{\mathrm{E}_{\mathbf{Top}}}{\hookrightarrow} \mathbf{LoATTB\text{-}Top}$, $\mathrm{E}_{\mathbf{Top}}((X_1, B_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, B_2, \tau_2)) = (X_1, \mathrm{E}_{\mathbf{B}}(B_1), \tau_1) \xrightarrow{(f,\mathrm{E}_{\mathbf{B}}(\varphi))} (X_2, \mathrm{E}_{\mathbf{B}}(B_2), \tau_2)$, which in general is not full (using the machinery of the proof of Proposition 2.5, $\mathrm{E}_{\mathbf{Top}}(\varnothing, B_1, B_1^{\varnothing}) \xrightarrow{(!,(\varphi,\psi))} \mathrm{E}_{\mathbf{Top}}(\varnothing, B_2, B_2^{\varnothing})$ is continuous, but never belongs to the image of $\mathrm{E}_{\mathbf{Top}}$). The new functor makes the diagram

$$\begin{array}{ccc}
\mathbf{LoB\text{-}Top} & \overset{\mathrm{E}_{\mathbf{Top}}}{\hookrightarrow} & \mathbf{LoATTB\text{-}Top} \\
{\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\
\mathbf{Set} \times \mathbf{LoB} & \underset{1_{\mathbf{Set}} \times \mathrm{E}_{\mathbf{B}}^{op}}{\hookrightarrow} & \mathbf{Set} \times \mathbf{LoATTB}
\end{array}$$

commute, showing that (in some cases) the category **LoATTB-Top** provides a proper extension of the category **LoB-Top**. Moreover, it appears that the former category induces another functor, which has more importance in the current developments. The new definition requires an additional (and very significant) notion related to catalg topology. This time, it is the concept of *topological system* introduced by S. Vickers [75] as a common framework for incorporating both topological spaces and their underlying algebraic structures – locales [35], thereby trying to merge point-set and pointless topology. Recently, the notion was successfully extended to include the case of lattice-valued topologies, the most significant results in the field achieved by J. T. Denniston, A. Melton, S. E. Rodabaugh [11, 12, 13, 14], C. Guido [25, 26] and S. Solovyov [66, 74].

Definition 2.9. Let $\mathbf{A}$ be a variety and let $\mathbf{C}$, $\mathbf{D}$ be subcategories of $\mathbf{LoA}$. A $(\mathbf{C},\mathbf{D})$-*topological system* ($(\mathbf{C},\mathbf{D})$-*system*) is a tuple $D = (\mathrm{pt}\,D, \Sigma D, \Omega D, \models)$, where $(\mathrm{pt}\,D, \Sigma D, \Omega D)$ is a $\mathbf{Set}\times\mathbf{C}\times\mathbf{D}$-object and $\mathrm{pt}\,D \times \Omega D \xrightarrow{\models} \Sigma D$ is a map (ΣD-*satisfaction relation on* $(\mathrm{pt}\,D, \Omega D)$) such that $\Omega D \xrightarrow{\models(x,-)} \Sigma D$ is an $\mathbf{A}$-homomorphism for every $x \in \mathrm{pt}\,D$. A $(\mathbf{C},\mathbf{D})$-*continuous map* $D_1 \xrightarrow{f} D_2$ is a $\mathbf{Set}\times\mathbf{C}\times\mathbf{D}$-morphism $(\mathrm{pt}\,D_1, \Sigma D_1, \Omega D_1) \xrightarrow{(\mathrm{pt}\,f,(\Sigma f)^{op},(\Omega f)^{op})} (\mathrm{pt}\,D_2, \Sigma D_2, \Omega D_2)$ such that for every $x \in \mathrm{pt}\,D_1$ and every $b \in \Omega D_2$, it follows that $\models_1(x, \Omega f(b)) = \Sigma f(\models_2(\mathrm{pt}\,f(x), b))$. $(\mathbf{C},\mathbf{D})$-$\mathbf{TopSys}$ is the category of $(\mathbf{C},\mathbf{D})$-systems and $(\mathbf{C},\mathbf{D})$-continuous maps, concrete over the product category $\mathbf{Set}\times\mathbf{C}\times\mathbf{D}$.

For the sake of shortness, the category $(\mathbf{LoA},\mathbf{LoA})$-$\mathbf{TopSys}$ is denoted $\mathbf{LoA}$-$\mathbf{TopSys}$, whereas the category $(\mathbf{S}_A,\mathbf{LoA})$-$\mathbf{TopSys}$ is denoted A-$\mathbf{TopSys}$. To provide the intuition for the concept, we list two important examples.

Example 2.10. $\mathbf{Loc}$-$\mathbf{TopSys}$ *is precisely the category of lattice-valued topological systems introduced by J. T. Denniston, A. Melton and S. E. Rodabaugh in* [12]. *Its subcategory* $\mathbf{2}$-$\mathbf{TopSys}$ *(*$\mathbf{2}$ *is the two-element frame* $\{\bot,\top\}$*) is isomorphic to the category* $\mathbf{TopSys}$ *of S. Vickers* [75].

Example 2.11. *Given a set* K*, the subcategory* K-$\mathbf{TopSys}$ *of* $\mathbf{LoSet}$-$\mathbf{TopSys}$ *is isomorphic to the category* $\mathsf{Chu}(\mathbf{Set}, K)$ *(or just* Chu_K*) comprising Chu spaces over a given set* K [5, 48]. *In particular,* $\mathsf{Chu}_{\mathbf{2}}$ *is the category* $\mathbf{Cont}$ *of* contexts *of formal concept analysis* [19, 76]*, and also the category* $\mathbf{IntSys}$ *of* interchange systems *introduced recently by J. T. Denniston, A. Melton and S. E. Rodabaugh* [13] *in connection with certain aspects of program semantics (the so-called* predicate transformers*) initiated by E. W. Dijkstra* [16]. *Sharing the same definition, the categories* $\mathsf{Chu}_{\mathbf{2}}$*,* $\mathbf{Cont}$ *and* $\mathbf{IntSys}$ *have quite different motivating theories.*

The framework of Definition 2.9 is closely related to the category $\mathbf{LoA}$-$\mathbf{Top}$, allowing the extension of the system *spatialization procedure*, introduced by S. Vickers [75] to extract their inherent topology.

Theorem 2.12.

(1) *There exists a full embedding* $\mathbf{LoA}\text{-}\mathbf{Top} \overset{\mathrm{E}}{\hookrightarrow} \mathbf{LoA}\text{-}\mathbf{TopSys}$ *defined by* $\mathrm{E}((X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)) = (X_1, A_1, \tau_1, \models_1) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (X_2, A_2, \tau_2, \models_2)$*, where* $\models_i(x, \alpha) = \alpha(x)$.
(2) *There exists a functor* $\mathbf{LoA}\text{-}\mathbf{TopSys} \xrightarrow{\mathrm{Spat}} \mathbf{LoA}\text{-}\mathbf{Top}$*, which is defined by the formula* $\mathrm{Spat}(D_1 \xrightarrow{f} D_2) = (\mathrm{pt}\,D_1, \Sigma D_1, \tau_1) \xrightarrow{(\mathrm{pt}\,f,(\Sigma f)^{op})} (\mathrm{pt}\,D_2, \Sigma D_2, \tau_2)$*, where* $\tau_j = \{\models_j(-, b) \mid b \in \Omega D_j\}$.
(3) Spat *is a right-adjoint-left-inverse to* E.
(4) *The category* $\mathbf{LoA}$-$\mathbf{Top}$ *is isomorphic to a full (regular mono)-coreflective subcategory of the category* $\mathbf{LoA}$-$\mathbf{TopSys}$.

The attentive reader has probably already guessed that the name "Spat" in the second item of Theorem 2.12 comes from "spatialization".

The new category of Definition 2.8 in hand, we can proceed to the definition of a new functor.

Proposition 2.13. *There is a functor* **LoATTB-Top** $\xrightarrow{\mathrm{E}_{\mathbf{ATT}}}$ **LoB-TopSys**, *which is given through the formula* $\mathrm{E}_{\mathbf{ATT}}((X_1, G_1, \tau_1) \xrightarrow{(f,g)} (X_2, G_2, \tau_2)) = (X_1 \times |\,\Omega\, G_1|, \Sigma\, G_1, \tau_1, \models_1) \xrightarrow{(f \times (\Omega\, g)^{*\,op}, (\Sigma\, g)^{op}, ((f,(\Omega\, g)^{op})^{\leftarrow})^{op})} (X_2 \times |\,\Omega\, G_2|, \Sigma\, G_2, \tau_2, \models_2)$*, where* $\models_i((x,b),\alpha) = (\Vdash_i(\alpha(x)))(b)$.

Proof. To show that the functor in question is correct on objects, notice that given $\lambda \in \Lambda_{\mathbf{B}}$ and $\alpha_i \in \tau$ for $i \in n_\lambda$, it follows that

$$\begin{aligned}\models((x,b),\omega^{\tau}_{\lambda}(\langle\alpha_i\rangle_{n_\lambda})) = (\Vdash((\omega^{\tau}_{\lambda}(\langle\alpha_i\rangle_{n_\lambda}))(x)))(b) = (\Vdash(\omega^{\Omega\, G}_{\lambda}(\langle\alpha_i(x)\rangle_{n_\lambda})))(b) = \\ (\omega^{(\Sigma\, G)^{|\,\Omega\, G|}}_{\lambda}(\langle\Vdash(\alpha_i(x))\rangle_{n_\lambda}))(b) = \omega^{\Sigma\, G}_{\lambda}(\langle(\Vdash(\alpha_i(x)))(b)\rangle_{n_\lambda}) = \\ \omega^{\Sigma\, G}_{\lambda}(\langle\models((x,b),\alpha_i)\rangle_{n_\lambda}).\end{aligned}$$

To check the preservation of continuity, use the fact that for $(x,b) \in X_1 \times |\,\Omega\, G_1|$ and $\alpha \in \tau_2$,

$$\begin{aligned}\models_1((x,b),(f,(\Omega\, g)^{op})^{\leftarrow}(\alpha)) = (\Vdash_1(((f,(\Omega\, g)^{op})^{\leftarrow}(\alpha))(x)))(b) = \\ (\Vdash_1(\Omega\, g \circ \alpha \circ f(x)))(b) = (\Sigma\, g \circ \Vdash_2(\alpha \circ f(x)))((\Omega\, g)^{*op}(b)) = \\ \Sigma\, g \circ \models_2((f(x),(\Omega\, g)^{*op}(b)),\alpha) = \Sigma\, g \circ \models_2(f \times (\Omega\, g)^{*op}(x,b),\alpha).\end{aligned}$$

□

It should be noticed at once that despite the notation, the functor of Proposition 2.13 never needs to be an embedding. In fact, the merits of the functor in question are highly dependant on the properties of the employed functor $\mathbf{B} \xrightarrow{(-)^*} \mathbf{Set}^{op}$. On the other hand, it is possible to restrict the domain of $\mathrm{E}_{\mathbf{ATT}}$ and obtain an embedding. Below we suggest two possible approaches, the first of which being rather straightforward.

Proposition 2.14. *Given a dual* **B**-*attachment* G *such that* $\Omega\, G$ *is non-empty, the restriction* G-**Top** $\xrightarrow{\mathrm{E}^G_{\mathbf{ATT}} = \mathrm{E}_{\mathbf{ATT}}|^{\Sigma G\text{-}\mathbf{TopSys}}_{G\text{-}\mathbf{Top}}} \Sigma\, G$-**TopSys** *is an embedding.*

Proof. Given a G-continuous map $(X_1,\tau_1) \xrightarrow{f} (X_2,\tau_2)$, $\mathrm{E}^G_{\mathbf{ATT}}((X_1,\tau_1) \xrightarrow{f} (X_2,\tau_2)) = (X_1 \times |\,\Omega\, G|, \tau_1, \models_1) \xrightarrow{(f \times 1_{|\,\Omega\, G|}, (f^{\leftarrow}_{\Omega\, G})^{op})} (X_2 \times |\,\Omega\, G|, \tau_2, \models_2)$ (recall our shortened notation for fixed-basis topological spaces) that implies the desired property, the condition on $\Omega\, G$ excluding the case of the constant functor mapping everything to the empty system. □

The second approach is more sophisticated. The restriction in question is provided by the concept of *stratified topological space* (the idea of stratification is due to R. Lowen [44], the term itself coined by P.-M. Pu and Y.-M. Liu [50]).

Definition 2.15. $\mathbf{LoATTB\text{-}Top}_{\varnothing k}$ is the full subcategory of **LoATTB-Top** of non-empty *stratified* spaces, i.e., spaces (X,G,τ) such that both X and $\Omega\, G$ are non-empty, and for every $a\in\Omega\, G$, the constant map a_X is in τ.

The notation "$(-)_k$" for stratified spaces comes from [51, 52] and is already widely accepted among the researchers, motivating us to follow their steps.

Proposition 2.16. *The restriction* $\mathbf{LoATTB\text{-}Top}_{\varnothing k}\xrightarrow{\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}=\mathrm{E}_{\mathbf{ATT}}|_{\mathbf{LoATTB\text{-}Top}_{\varnothing k}}}$ **LoB-TopSys** *provides an embedding.*

Proof. Let $(X_1,G_1,\tau_1)\overset{(f_1,g_1)}{\underset{(f_2,g_2)}{\rightrightarrows}}(X_2,G_2,\tau_2)$ be a pair of $\mathbf{LoATTB\text{-}Top}_{\varnothing k}$-morphisms. To show that $\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}$ embeds objects, notice that $\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}(X_1,G_1,\tau_1)=\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}(X_2,G_2,\tau_2)$ implies $X_1\times|\,\Omega\, G_1|=X_2\times|\,\Omega\, G_2|$, $\Sigma\, G_1=\Sigma\, G=\Sigma\, G_2$, $\tau_1=\tau=\tau_2$ and $(X_1\times|\,\Omega\, G_1|)\times\tau\xrightarrow{\models_1}\Sigma\, G=(X_2\times|\,\Omega\, G_2|)\times\tau\xrightarrow{\models_2}\Sigma\, G$. The assumption on non-emptiness (which can not be avoided) provides $X_1=X=X_2$ and $|\,\Omega\, G_1|=Y=|\,\Omega\, G_2|$. To show that $\Omega\, G_1=\Omega\, G_2$, take some $x_0\in X$ and then, given $\lambda\in\Lambda_{\mathbf{B}}$ and $b_i\in Y$ for $i\in n_\lambda$, $\omega^{\Omega\, G_1}_\lambda(\langle b_i\rangle_{n_\lambda})=(\omega^{\tau_1}_\lambda(\langle \underline{b_i}\rangle_{n_\lambda}))(x_0)=(\omega^{\tau_2}_\lambda(\langle \underline{b_i}\rangle_{n_\lambda}))(x_0)=\omega^{\Omega\, G_2}_\lambda(\langle b_i\rangle_{n_\lambda})$, implying $\Omega\, G_1=\Omega\, G=\Omega\, G_2$. To show that $\Vdash_1=\Vdash_2$, employ the existing x_0 to get that for every $b_1,b_2\in\Omega\, G$, $(\Vdash_1(b_1))(b_2)=(\Vdash_1(\underline{b_1}(x_0)))(b_2)=\models_1((x_0,b_2),\underline{b_1})=\models_2((x_0,b_2),\underline{b_1})=(\Vdash_2(b_1))(b_2)$.

To show faithfulness of $\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}$, use the fact that $\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}(f_1,g_1)=\mathrm{E}^{\varnothing k}_{\mathbf{ATT}}(f_2,g_2)$ implies $f_1\times(\Omega\, g_1)^{*op}=f_2\times(\Omega\, g_2)^{*op}$, $\Sigma\, g_1=\Sigma\, g=\Sigma\, g_2$ and $(f_1,(\Omega\, g_1)^{op})^{\leftarrow}=(f_2,(\Omega\, g_2)^{op})^{\leftarrow}$. The non-emptiness requirement provides $f_1=f=f_2$ and also $(\Omega\, g_1)^{*op}=(\Omega\, g_2)^{*op}$. To verify that $\Omega\, g_1=\Omega\, g_2$, use the fact that given $b\in\Omega\, G_2$, $(\Omega\, g_1)(b)=(\Omega\, g_1\circ\underline{b}\circ f)(x_0)=((f,(\Omega\, g_1)^{op})^{\leftarrow}(\underline{b}))(x_0)=((f,(\Omega\, g_2)^{op})^{\leftarrow}(\underline{b}))(x_0)=(\Omega\, g_2)(b)$. □

At the end of this section, we finally define the main object of our interest, namely, a particular functor. It provides an analogue of the functor H, introduced in [73] as a generalization of the functor L-**Top** $\xrightarrow{(-)^{\star}}$ **Top** of [26] (already mentioned in Introduction), with the aim to produce a convenient framework for studying categorical properties of the hypergraph functors. It is one of the main goals of this paper to explore the nature of the new functor and its relationships to its predecessors.

Definition 2.17. There exists a functor **LoATTB-Top** $\xrightarrow{\mathrm{H}_{\mathbf{ATT}}}$ **LoB-Top** $=$ **LoATTB-Top** $\xrightarrow{\mathrm{E}_{\mathbf{ATT}}}$ **LoB-TopSys** $\xrightarrow{\mathrm{Spat}}$ **LoB-Top**, $\mathrm{H}_{\mathbf{ATT}}((X_1,G_1,\tau_1)\xrightarrow{(f,g)}(X_2,G_2,\tau_2))=(X_1\times|\,\Omega\, G_1|,\Sigma\, G_1,\tilde{\tau}_1)\xrightarrow{(f\times(\Omega\, g)^{*op},(\Sigma\, g)^{op})}(X_2\times|\,\Omega\, G_2|,\Sigma\, G_2,\tilde{\tau}_2)$, where $\tilde{\tau}_i=\{\tilde{\alpha}=(\Vdash_i(\alpha(-)))(-)\,|\,\alpha\in\tau_i\}$. Given a **LoATTB**-object G, the respective *fixed-basis* functor G-**Top** $\xrightarrow{\mathrm{H}^G_{\mathbf{ATT}}}$ $\Sigma\, G$-**Top** is defined by the formula $\mathrm{H}^G_{\mathbf{ATT}}((X_1,\tau_1)\xrightarrow{f}(X_2,\tau_2))=(X_1\times|\,\Omega\, G|,\tilde{\tau}_1)\xrightarrow{f\times 1_{|\,\Omega\, G|}}(X_2\times|\,\Omega\, G|,\tilde{\tau}_2)$.

Unlike [73], we are not going to touch the topic of hypergraph functors in this paper, restricting our attention to the functor $\mathrm{H}_{\mathbf{ATT}}$ itself. We begin with the remark that certain properties of the dual attachment G can help to provide an embedding property for the resulting functor $\mathrm{H}^{G}_{\mathbf{ATT}}$.

Definition 2.18. A dual attachment G is called

(1) Ω-*spatial* provided that for every $b_1, b_2 \in \Omega\, G$ such that $b_1 \neq b_2$, $\Vdash(b_1) \neq \Vdash(b_2)$ ($\Vdash$ is injective).
(2) Σ-*spatial* provided that for every $b_1, b_2 \in \Omega\, G$ such that $b_1 \neq b_2$, there exists some $b \in \Omega\, G$ such that $(\Vdash(b))(b_1) \neq (\Vdash(b))(b_2)$.

After brief consideration, the reader will easily see that Ω-spatiality and Σ-spatiality are quite different notions. A nice example on the topic is provided by the category **ATTSet**. The map $\mathbb{I} \xrightarrow{\Vdash} \mathbb{I}^{\mathbb{I}}$, taking every $a \in \mathbb{I}$ to the identity $1_{\mathbb{I}}$, gives a dual attachment $G = (\mathbb{I}, \mathbb{I}, \Vdash)$, which is Σ-spatial but not Ω-spatial. On the other hand, changing the definition to take every $a \in \mathbb{I}$ to the constant map $a_{\mathbb{I}}$, provides an attachment which is Ω-spatial but not Σ-spatial. Examples for more complicated varieties can be found in [73].

Proposition 2.19. *Given an Ω-spatial attachment G with the property that $\Omega\, G$ is non-empty, the functor G-$\mathbf{Top} \xrightarrow{\mathrm{H}^{G}_{\mathbf{ATT}}} \Sigma\, G$-$\mathbf{Top}$ is an embedding.*

Proof. It will be enough to show the injectivity on objects. Given two spaces (X_1, τ_1) and (X_2, τ_2) such that $\mathrm{H}^{G}_{\mathbf{ATT}}(X_1, \tau_1) = \mathrm{H}^{G}_{\mathbf{ATT}}(X_2, \tau_2)$, the non-emptiness of $\Omega\, G$ implies $X_1 = X = X_2$. To show that $\tau_1 = \tau_2$, notice that given $\alpha_1 \in \tau_1$, $\tilde{\alpha}_1 \in \tilde{\tau}_1 = \tilde{\tau}_2$ and, therefore, $\tilde{\alpha}_1 = \tilde{\alpha}_2$ for some $\alpha_2 \in \tau_2$. Given $x \in X$, $(\Vdash_1(\alpha_1(x)))(b) = \tilde{\alpha}_1(x, b) = \tilde{\alpha}_2(x, b) = (\Vdash_2(\alpha_2(x)))(b)$ for every $b \in \Omega\, G$ and that implies $\alpha_1(x) = \alpha_2(x)$ by Ω-spatiality of G. As a result, $\alpha_1 = \alpha_2$, providing $\tau_1 \subseteq \tau_2$. The converse inclusion is similar. □

3. Dual attachment pairs

This section clarifies the word "dual" in the term "dual attachment" used in this paper. The motivation for the choice comes from a particular property of attachment found out in [26], namely, the existence of a functor L-$\mathbf{Top} \xrightarrow{(-)^{\star}} \mathbf{Top}$ (already mentioned in Introduction), which takes an L-topological space (X, τ) to the crisp space $(S_X, \tau^{\star})$, where S_X is the set of L-points of X, and $\tau^{\star}$ consists of the sets $\alpha^{\star}$ for every $\alpha \in \tau$, comprising precisely those L-points, which are attached to the particular α in question. A catalg analogue of the above-mentioned functor has been already considered in [73], whose counterpart for the current setting is given in Definition 2.17. It is the main purpose of this section to show that both functors coincide in case the respective attachments form a *dual attachment pair*. We begin with the definition of a generalized version of the above-mentioned functors, which require some additional preliminaries contained in the following definition and proposition.

Definition 3.1. Let $\mathbf{A}$ be a variety and let $(\|-\|, \mathbf{B})$ be a reduct of $\mathbf{A}$. A dual $\mathbf{B}$-attachment G is called $\mathbf{A}$-*derived* provided that there exist $\mathbf{A}$-algebras A_Ω, A_Σ such that $\Omega\, G = \|A_\Omega\|$, $\Sigma\, G = \|A_\Sigma\|$.

Proposition 3.2. *Let $\mathbf{A}$ be a variety, let $(\|-\|, \mathbf{B})$ be a reduct of $\mathbf{A}$ and let A be an $\mathbf{A}$-algebra. There exist two functors:*

(1) $\|A\|$-$\mathbf{Top} \xrightarrow{\mathrm{Xt}_{\|A\|A}} A$-$\mathbf{Top}$ *defined by* $\mathrm{Xt}_{\|A\|A}((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, \langle\tau_1\rangle) \xrightarrow{f} (X_2, \langle\tau_2\rangle)$ *with* $\langle\tau_i\rangle$ *the* $\mathbf{A}$*-subalgebra of* A *generated by* τ_i*;*
(2) A-$\mathbf{Top} \xrightarrow{\mathrm{Rd}_{A\|A\|}} \|A\|$-$\mathbf{Top}$ *defined by* $\mathrm{Rd}_{A\|A\|}((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, \|\tau_1\|) \xrightarrow{f} (X_2, \|\tau_2\|)$.

If G is an $\mathbf{A}$-derived dual $\mathbf{B}$-attachment, then there is a functor A_Ω-$\mathbf{Top} \xrightarrow{\mathcal{H}^G_{\mathbf{ATT}}} A_\Sigma$-$\mathbf{Top}$ *defined by commutativity of the following diagram:*

$$\begin{array}{ccc} A_\Omega\text{-}\mathbf{Top} & \overset{\mathcal{H}^G_{\mathbf{ATT}}}{\dashrightarrow} & A_\Sigma\text{-}\mathbf{Top} \\ {\scriptstyle \mathrm{Rd}_{A_\Omega\,\Omega\,G}}\downarrow & & \uparrow{\scriptstyle \mathrm{Xt}_{\Sigma\,G A_\Sigma}} \\ G\text{-}\mathbf{Top} & \underset{\mathrm{H}^G_{\mathbf{ATT}}}{\longrightarrow} & \Sigma\,G\text{-}\mathbf{Top}. \end{array}$$

Proof. It is enough to verify that the functor $\mathrm{Xt}_{\|A\|A}$ preserves continuity. We use the simple fact that given a homomorphism $A_1 \xrightarrow{\varphi} A_2$ and a subset $S \subseteq A_1$, $\varphi^{\rightarrow}(\langle S\rangle) = \langle\varphi^{\rightarrow}(S)\rangle$ [70]. As a result, $(f^{\leftarrow}_A)^{\rightarrow}(\langle\tau_2\rangle) = \langle(f^{\leftarrow}_A)^{\rightarrow}(\tau_2)\rangle \subseteq \langle\tau_1\rangle$. ☐

The reader should notice that "Xt" (resp. "Rd") is the abbreviation for "extension" (resp. "reduction"), used to underline the action of the functor in question, i.e., to extend (resp. reduce) the algebraic structure. Both functors will play an important role in the subsequent developments.

To compare the new functor with the already existing setting of [73], one should recall some results from its approach to the concept of attachment.

Proposition 3.3. *There exists a functor* $\mathbf{LoAttA}$-$\mathbf{Top} \xrightarrow{\mathrm{E}_{\mathbf{Att}}} \mathbf{LoA}$-$\mathbf{TopSys}$, *which is given by the formula* $\mathrm{E}_{\mathbf{Att}}((X_1, F_1, \tau_1) \xrightarrow{(f,g)} (X_2, F_2, \tau_2)) = (X_1 \times |\,\Omega\, F_1|, \Sigma\, F_1, \tau_1, \models_1) \xrightarrow{(f\times(\Omega\, g)^{*\,op}, (\Sigma\, g)^{op}, ((f, (\Omega\, g)^{op})^{\leftarrow})^{op})} (X_2 \times |\,\Omega\, F_2|, \Sigma\, F_2, \tau_2, \models_2)$, $\models_i((x,a),\alpha) = (\Vdash_i(a))(\alpha(x))$.

The reader is advised to pay attention to the important fact that the only difference in the definition of the functors of Propositions 2.13, 3.3 concerns the respective satisfaction relation.

Definition 3.4. There exists a functor $\mathbf{LoAttA}$-$\mathbf{Top} \xrightarrow{\mathrm{H}_{\mathbf{Att}}} \mathbf{LoA}$-$\mathbf{Top}$ $=$ $\mathbf{LoAttA}$-$\mathbf{Top} \xrightarrow{\mathrm{E}_{\mathbf{Att}}} \mathbf{LoA}$-$\mathbf{TopSys} \xrightarrow{\mathrm{Spat}} \mathbf{LoA}$-$\mathbf{Top}$, $\mathrm{H}_{\mathbf{Att}}((X_1, F_1, \tau_1) \xrightarrow{(f,g)} (X_2, F_2, \tau_2)) = (X_1 \times |\,\Omega\, F_1|, \Sigma\, F_1, \hat{\tau}_1) \xrightarrow{(f\times(\Omega\, g)^{*\,op}, (\Sigma\, g)^{op})} (X_2 \times |\,\Omega\, F_2|, \Sigma\, F_2, \hat{\tau}_2)$, where $\hat{\tau}_i = \{\hat{\alpha} = (\Vdash_i(-))(\alpha(-)) \,|\, \alpha \in \tau_i\}$. Given a **LoAttA**-object F, the respective *fixed-basis* functor F-**Top** $\xrightarrow{\mathrm{H}^F_{\mathbf{Att}}}$ $\Sigma\, F$-**Top** is given by $\mathrm{H}^F_{\mathbf{Att}}((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1 \times |\,\Omega\, F|, \hat{\tau}_1) \xrightarrow{f \times 1_{|\,\Omega\, F|}} (X_2 \times |\,\Omega\, F|, \hat{\tau}_2)$.

By analogy with Proposition 3.2(3), one obtains the functor A_Ω-**Top** $\xrightarrow{\mathcal{H}^F_{\mathbf{Att}}}$ A_Σ-**Top** (notice the difference in the notations). The crucial question arises on when the two functors $\mathcal{H}^F_{\mathbf{Att}}$ and $\mathcal{H}^G_{\mathbf{ATT}}$ coincide, and that is precisely the point for dual attachment pairs to come in play.

Definition 3.5. Let **A** be a variety, let $(\|-\|, \mathbf{B})$ and $(\|-\|, \mathbf{C})$ be reducts of **A**, and let F and G be a **B**-attachment and a dual **C**-attachment respectively.

(1) Both F and G are called *reduct attachments*.
(2) The pair (F, G) is an *attachment pair w.r.t.* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.
(3) (F, G) is a *related attachment pair* provided that both F and G are **A**-derived, and $A^F_\Omega = A_\Omega = A^G_\Omega$, $A^F_\Sigma = A_\Sigma = A^G_\Sigma$.
(4) (F, G) is a *dual attachment pair* provided that (F, G) is a related attachment pair, and for every $a_1, a_2 \in A_\Omega$, $(\Vdash(a_1))(a_2) = (\Vvdash(a_2))(a_1)$ w.r.t. the maps $|A_\Omega| \underset{\Vvdash}{\overset{\Vdash}{\rightrightarrows}} \mathbf{Set}(|A_\Omega|, |A_\Sigma|)$.

Every related attachment pair gives two functors A_Ω-**Top** $\underset{\mathcal{H}^G_{\mathbf{ATT}}}{\overset{\mathcal{H}^F_{\mathbf{Att}}}{\rightrightarrows}}$ A_Σ-**Top**. With a dual attachment pair in hand, these two functors coincide.

Proposition 3.6. *Given a dual attachment pair* (F, G), $\mathcal{H}^F_{\mathbf{Att}} = \mathcal{H}^G_{\mathbf{ATT}}$.

Proof. Since the case of morphisms is clear, it will be enough to show equality of the functors on objects. Given an A_Ω-space (X, τ), it is sufficient to verify that $\langle\hat{\tau}\rangle = \langle\tilde{\tau}\rangle$. Given $\alpha \in \tau$, $\hat{\alpha}(x, a) = ((\Vdash(-))(\alpha(-)))(x, a) = (\Vdash(a))(\alpha(x)) = (\Vvdash(\alpha(x)))(a) = ((\Vvdash(\alpha(-)))(-))(x, a) = \tilde{\alpha}(x, a)$ for every $(x, a) \in X \times |A_\Omega|$. Thus $\hat{\tau} = \tilde{\tau}$, implying $\langle\hat{\tau}\rangle = \langle\tilde{\tau}\rangle$. □

A good illustration of Proposition 3.6 is provided by Observation 2 from Introduction, which gives a dual attachment pair (F, G) w.r.t. the varieties $(\mathbf{CLat}, \mathbf{QFrm}, \mathbf{QFrm})$, where $F = (L, \mathbf{2}, \Vdash)$ is based on a complete chain L and the map

$$(\Vdash(a_1))(a_2) = \begin{cases} \top, & a_1 < a_2, \\ \bot, & \text{otherwise.} \end{cases}$$

To get more intuition for the attachment pair (F, G), one can represent every map $\Vdash(a)$ as a particular subset of L (the preimage of $\{\top\}$ under the map in question), i.e., $\Vdash(a) = \{b \in L \,|\, a < b\} = \upharpoonleft a$ (notice that $\Vdash(\top) = \varnothing$). The dual attachment $\Vvdash$ is then the collection of sets $\Vvdash(a) = \{b \in L \,|\, b < a\} = \downharpoonleft a$ ($\Vvdash(\bot) = \varnothing$ now) for every $a \in L$. It is easy to see that in this particular case, the attachment duality reduces to the usual, well-known in mathematics, order-theoretic duality. The main point of Proposition 3.6 is that the functors

L-$\mathbf{Top} \underset{\mathcal{H}^G_{\mathbf{ATT}}}{\overset{\mathcal{H}^F_{\mathbf{Att}}}{\rightrightarrows}} (\mathbf{2}\text{-}\mathbf{Top} \cong \mathbf{Top})$ generated by F and G coincide. On the other hand, straightforward computations backed by [73] show that $\mathcal{H}^F_{\mathbf{Att}}$ is precisely the functor L-$\mathbf{Top} \xrightarrow{(-)^\star} \mathbf{Top}$ of [26]. It follows that a two-fold representation of the already well-known notion is obtained. The reader should pay attention to the fact that the case $L = \mathbf{2}$ does not result in the identity functor on $\mathbf{Top}$, since a topological space (X, τ) is taken to the space $(X \times |\mathbf{2}|, \langle\|\hat{\tau}\|\rangle = \langle\|\tilde{\tau}\|\rangle)$, which has a different carrier set.

The reader will easily find other examples on the topic. It is important to underline, however, that the case of an unrelated attachment pair can provide completely different (incomparable) functors $\mathcal{H}^F_{\mathbf{Att}}$, $\mathcal{H}^G_{\mathbf{ATT}}$.

4. Existence of dual attachment pairs

The previous section showed that the case of a dual attachment pair has the crucial property of equality of the derived functors. On the other hand, the functors of a just related attachment pair need not coincide which gives dual attachment pairs even more importance. A natural question on the existence of dual attachment pairs arises. This short section clarifies the situation.

Proposition 4.1. *Let F be a $\mathbf{B}$-attachment which is a reduct attachment w.r.t. $\mathbf{A}$. There exists a dual $\mathbf{C}$-attachment G such that (F, G) is a dual attachment pair iff the following conditions are fulfilled:*

1. *F is $\mathbf{A}$-derived ($\Omega\, F = \|A_\Omega\|$ and $\Sigma\, F = \|A_\Sigma\|$);*
2. *there exists a reduct $(\|-\|, \mathbf{C})$ of $\mathbf{A}$ such that $\|A_\Omega\| \xrightarrow{\Vdash} \mathbf{C}(\|A_\Omega\|, \|A_\Sigma\|)$.*

Proof. For the necessity, notice that, firstly, $A^F_\Omega = A_\Omega = A^G_\Omega$, $A^F_\Sigma = A_\Sigma = A^G_\Sigma$, and, secondly, given $\lambda \in \Lambda_{\mathbf{C}}$ and $a_i \in \|A_\Omega\|$ for $i \in n_\lambda$,

$$(\Vdash(a))(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})) = (\Vvdash(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})))(a) =$$
$$(\omega^{\|A_\Sigma\|^{|A_\Omega|}}_\lambda(\langle\Vvdash(a_i)\rangle_{n_\lambda}))(a) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vvdash(a_i))(a)\rangle_{n_\lambda}) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vdash(a))(a_i)\rangle_{n_\lambda})$$

for every $a \in \|A_\Omega\|$. The sufficiency is slightly more sophisticated. Define the required dual attachment G by $\Omega\, G = \|A_\Omega\|$, $\Sigma\, G = \|A_\Sigma\|$ (the respective reducts are taken in the variety $\mathbf{C}$) together with $\Omega\, G \xrightarrow{\Vvdash} (\Sigma\, G)^{|\,\Omega\, G|}$ given by $(\Vvdash(a_1))(a_2) = (\Vdash(a_2))(a_1)$. The only challenge now is to show that $\Vvdash$ is a $\mathbf{C}$-homomorphism. Given $\lambda \in \Lambda_{\mathbf{C}}$ and $a_i \in \Omega\, G$ for $i \in n_\lambda$, $(\Vvdash(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})))(a) = (\Vdash(a))(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vdash(a))(a_i)\rangle_{n_\lambda}) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vvdash(a_i))(a)\rangle_{n_\lambda}) = (\omega^{\|A_\Sigma\|^{|A_\Omega|}}_\lambda(\langle a_i\rangle_{n_\lambda}))(a)$ for every $a \in \Omega\, G$. □

An example for the proposition is provided by the dual attachment pair (F, G) with $F = (L, \mathbf{2}, \Vdash)$, mentioned at the end of the previous section, the second of the requirements (the first being obvious) verified in Introduction as follows: $(\Vdash(a))(\bigvee S) = \top$ iff $a < \bigvee S$ iff $a < s$ for some $s \in S$ iff $(\Vdash(a))(s) = \top$ for some $s \in S$ iff $(\bigvee_{s \in S} \Vdash(a))(s) = \top$, whereas $(\Vdash(a))(s \wedge t) = \top$ iff $a < s \wedge t$

iff $a < s$ and $a < t$ iff $(\Vdash(a))(s) \wedge (\Vdash(a))(t) = \top$. The converse way (from dual attachment to attachment) is equally easy and can be run through as follows.

Proposition 4.2. *Let G be a dual $\mathbf{C}$-attachment which is a reduct attachment w.r.t. $\mathbf{A}$. There exists a $\mathbf{B}$-attachment F such that (F, G) is a dual attachment pair iff the following conditions are fulfilled:*

1. *G is $\mathbf{A}$-derived ($\Omega\, G = \|A_\Omega\|$ and $\Sigma\, G = \|A_\Sigma\|$);*
2. *there exists a reduct $(\|-\|, \mathbf{B})$ of $\mathbf{A}$ such that $\|A_\Omega\| \xrightarrow{\Vvdash} \|A_\Sigma\|^{|A_\Omega|}$ is a $\mathbf{B}$-homomorphism.*

Proof. For the necessity notice that, firstly, $A^G_\Omega = A_\Omega = A^F_\Omega$, $A^G_\Sigma = A_\Sigma = A^F_\Sigma$, and, secondly, given $\lambda \in \Lambda_{\mathbf{B}}$ and $a_i \in \|A_\Omega\|$ for $i \in n_\lambda$,

$$((\Vvdash(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})))(a) = (\Vdash(a))(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vdash(a))(a_i)\rangle_{n_\lambda}) =$$
$$\omega^{\|A_\Sigma\|}_\lambda(\langle(\Vvdash(a_i))(a)\rangle_{n_\lambda}) = (\omega^{\|A_\Sigma\|^{|A_\Omega|}}_\lambda(\langle \Vvdash(a_i)\rangle_{n_\lambda}))(a)$$

for every $a \in \|A_\Omega\|$. To show the sufficiency, define an attachment F by $\Omega\, F = \|A_\Omega\|$, $\Sigma\, F = \|A_\Sigma\|$ together with $\Omega\, F \xrightarrow{\Vdash} \mathbf{B}(\Omega\, F, \Sigma\, F)$ given by $(\Vdash(a_1))(a_2) = (\Vvdash(a_2))(a_1)$. It should be verified that $\Vdash(a)$ is a $\mathbf{B}$-homomorphism for every $a \in \Omega\, F$. Given $\lambda \in \Lambda_{\mathbf{B}}$ and $a_i \in \Omega\, F$ for $i \in n_\lambda$,

$$(\Vdash(a))(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})) = (\Vvdash(\omega^{\|A_\Omega\|}_\lambda(\langle a_i\rangle_{n_\lambda})))(a) =$$
$$(\omega^{\|A_\Sigma\|^{|A_\Omega|}}_\lambda(\langle \Vvdash(a_i)\rangle_{n_\lambda}))(a) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vvdash(a_i))(a)\rangle_{n_\lambda}) = \omega^{\|A_\Sigma\|}_\lambda(\langle(\Vdash(a))(a_i)\rangle_{n_\lambda}).$$

□

We close this section with the remark (already mentioned in Introduction) that the concept of duality for attachment used in this paper is developed in the framework of arbitrary algebras (possibly) void of any kind of order relation and, therefore, our current setting is *not* the duality induced by a partial order. On the other hand, the approach is *neither* a duality of category theory, since the underlying category of the respective attachment is not dualized. Based on the employed framework of catalg topology, the type of duality presented in the paper could be called categorically-algebraic. It will be the topic of our forthcoming papers to find the proper place of the new notion in mathematics. The current manuscript will continue exploring another aspects of attachment.

5. Natural transformations induced by attachment

It was shown in [26] that every attachment $\mathcal{A}$ in a complete lattice L provides a frame homomorphism $L^X \xrightarrow{(-)^\star} \mathcal{P}(S_X)$ for every set X (simply taking $\alpha \in L^X$ to the set $\alpha^\star$ of all L-points attached to α), which gives rise to the already mentioned functor L-**Top** $\xrightarrow{(-)^\star}$ **Top**. This section extends the map to our catalg setting, resulting in consequences important for the whole development.

Proposition 5.1. *Let F be a $\mathbf{B}$-attachment and let G be a dual $\mathbf{C}$-attachment. For every set X, there exist a $\mathbf{B}$-homomorphism $(\Omega\, F)^X \xrightarrow{(-)^{\Vdash}} (\Sigma\, F)^{X \times |\,\Omega\, F|}$*

defined by $\alpha^{\Vdash}(x,b) = (\Vdash(b))(\alpha(x))$, *and a* $\mathbf{C}$*-homomorphism* $(\Omega\, G)^X \xrightarrow{(-)^{\Vvdash}} (\Sigma\, G)^{X \times |\,\Omega\, G|}$ *defined by* $\alpha^{\Vvdash}(x,b) = (\Vvdash(\alpha(x)))(b)$. *If* (F,G) *is a related attachment pair, then the maps have the same (co)domain* $|A_\Omega|^X \underset{(-)^{\Vdash}}{\overset{(-)^{\Vvdash}}{\rightrightarrows}} |A_\Sigma|^{X \times |A_\Omega|}$. *If* (F,G) *is a dual attachment pair, then the maps coincide.*

Proof. To prove the first claim, we show that the map $(-)^{\Vvdash}$ provides a $\mathbf{C}$-homomorphism. Given $\lambda \in \Lambda_{\mathbf{C}}$ and $\langle \alpha_i \rangle_{n_\lambda} \in (\Omega\, G)^X$,

$$(\omega_\lambda^{(\Omega\, G)^X}(\langle \alpha_i \rangle_{n_\lambda}))^{\Vvdash}(x,b) = (\Vvdash((\omega_\lambda^{(\Omega\, G)^X}(\langle \alpha_i \rangle_{n_\lambda}))(x)))(b) = (\Vvdash(\omega_\lambda^{\Omega\, G}(\langle \alpha_i(x) \rangle_{n_\lambda})))(b) = (\omega_\lambda^{(\Sigma\, G)^{|\,\Omega\, G|}}(\langle \Vvdash(\alpha_i(x)) \rangle_{n_\lambda}))(b) = \omega_\lambda^{\Sigma\, G}(\langle (\Vvdash(\alpha_i(x)))(b) \rangle_{n_\lambda}) = \omega_\lambda^{\Sigma\, G}(\langle \alpha_i^{\Vvdash}(x,b) \rangle_{n_\lambda}) = (\omega_\lambda^{(\Sigma\, G)^{X \times |\,\Omega\, G|}}(\langle \alpha_i^{\Vvdash} \rangle_{n_\lambda}))(x,b)$$

for every $(x,b) \in X \times |\,\Omega\, G|$. The second claim is obvious. For the last statement, notice that given $\alpha \in |A_\Omega|^X$, $\alpha^{\Vvdash}(x,a) = (\Vvdash(\alpha(x)))(a) = (\Vdash(a))(\alpha(x)) = \alpha^{\Vdash}(x,a)$ for every $(x,a) \in X \times |A_\Omega|$. □

After a closer scrutiny, it appears that the homomorphisms of Proposition 5.1 are actually components of natural transformations (the fact, never mentioned in [26]). To prove the claim, start with the preliminary remark that given a dual $\mathbf{C}$-attachment G, there exists a functor $\mathbf{Set} \xrightarrow{(-\times|\,\Omega\, G|)^{\leftarrow}_{\Sigma\, G}} \mathbf{LoC}$ defined by commutativity of the following triangle:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{-\times|\,\Omega\, G|} & \mathbf{Set} \\ & \searrow^{(-\times|\,\Omega\, G|)^{\leftarrow}_{\Sigma\, G}} & \downarrow{\scriptstyle (-)^{\leftarrow}_{\Sigma\, G}} \\ & & \mathbf{LoC}, \end{array}$$

where $-\times|\,\Omega\, G|$ is the standard *product functor* [1] defined by the formula $(-\times|\,\Omega\, G|)(X_1 \xrightarrow{f} X_2) = X_1 \times |\,\Omega\, G| \xrightarrow{f \times 1_{|\,\Omega\, G|}} X_2 \times |\,\Omega\, G|$.

Proposition 5.2. *Every dual* $\mathbf{C}$*-attachment* G *provides a natural transformation* $(-\times|\,\Omega\, G|)^{\leftarrow}_{\Sigma\, G} \xrightarrow{((-)^{\Vvdash})^{op}} (-)^{\leftarrow}_{\Omega\, G}$.

Proof. Given a map $X_1 \xrightarrow{f} X_2$, one has to verify commutativity of the diagram

$$\begin{array}{ccc} (\Omega\, G)^{X_2} & \xrightarrow{(-)^{\Vvdash}_{X_2}} & (\Sigma\, G)^{X_2 \times |\,\Omega\, G|} \\ {\scriptstyle f^{\leftarrow}_{\Omega\, G}}\downarrow & & \downarrow{\scriptstyle (f \times 1_{|\,\Omega\, G|})^{\leftarrow}_{\Sigma\, G}} \\ (\Omega\, G)^{X_1} & \xrightarrow[(-)^{\Vvdash}_{X_1}]{} & (\Sigma\, G)^{X_1 \times |\,\Omega\, G|}, \end{array}$$

and that follows from the fact that $(((-)^{\Vvdash}_{X_1} \circ f^{\leftarrow}_{\Omega\, G})(\alpha))(x,b) = (\alpha \circ f)^{\Vvdash}(x,b) = (\Vvdash(\alpha \circ f(x)))(b) = \alpha^{\Vvdash}(f(x),b) = (\alpha^{\Vvdash} \circ (f \times 1_{|\,\Omega\, G|}))(x,b) = (((f \times 1_{|\,\Omega\, G|})^{\leftarrow}_{\Sigma\, G} \circ (-)^{\Vvdash}_{X_2})(\alpha))(x,b)$ for every $\alpha \in (\Omega\, G)^{X_2}$ and every $(x,b) \in X_1 \times |\,\Omega\, G|$. □

Similarly, one gets a natural transformation $(- \times |\,\Omega\, F|)^{\leftarrow}_{\Sigma\, F} \xrightarrow{((-)^{\Vdash})^{op}} (-)^{\leftarrow}_{\Omega\, F}$ for a given **B**-attachment F, which reduces to the setting of [26] for **B** being the variety **Frm** of frames. In case of a dual attachment pair, (as one might expect) both natural transformations coincide. To generalize the passage of [26] from a natural transformation to a functor, we need some additional notions from the realm of catalg topology, presented briefly in the subsequent section.

6. Categorically-algebraic topology and attachment

In this section we recall from [71] basic concepts of *categorically-algebraic* (*catalg*) *topology* (see also [64, 66, 72]), which bring to light the crucial property of attachment, i.e., generation of a functor between two topological settings. The approach is motivated by the currently dominating in the fuzzy community *point-set lattice-theoretic* (*poslat*) *topology* introduced by S. E. Rodabaugh [55] and developed by P. Eklund, C. Guido, U. Höhle, T. Kubiak, A. Šostak and the initiator himself [17, 24, 33, 39, 40, 57]. The main advantage of the new setting is the fact that apart from incorporating as special subcases the most important approaches to (lattice-valued) topology and providing convenient means of interaction between them, the catalg framework ultimately erases the border between crisp and many-valued developments, producing a theory which underlines the algebraic essence of the whole (not only lattice-valued) mathematics, thereby propagating algebra as the main driving force of modern exact sciences. It should be noticed immediately that some parts of the theory have already been used throughout the paper (Definition 2.7). The current section provides a more rigid foundation for the approach and backs it by several motivating examples, to give the flavor of fruitfulness of the new theory.

The setting is based in a mixture of *powerset theories* of [61, Definition 3.5] (see also [60, 62]) and *topological theories* of [1, Exercise 22B].

Definition 6.1. A *variety-based backward powerset theory* (*vbp-theory*) in a category **X** (the *ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$ to the dual of a variety **A**.

To get the intuition for the concept, the reader is advised to recall the functor of Proposition 2.6, providing the main example for the notion and incorporating many approaches to powerset operators popular in lattice-valued mathematics.

Example 6.2.

(1) $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}} \mathbf{LoCBool}$, *where* **CBool** *is the variety of* complete Boolean algebras *(complete, complemented, distributive lattices) and* $\mathbf{2} = \{\bot, \top\}$*, provides the standard preimage operator, mentioned before Proposition 2.6.*

(2) $\mathbf{Set} \times \mathbf{S}_{\mathbb{I}} \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_{\mathbb{I}}} \mathbf{DmLoc}$ *(cf. Definition 2.2), where* $\mathbb{I} = [0, 1]$ *is the unit interval, gives the fixed-basis fuzzy approach of L. A. Zadeh* [77].

(3) $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_1=(-)^{\leftarrow}_L} \mathbf{Loc}$ *provides the fixed-basis L-fuzzy approach of J. A. Goguen* [21]. *The setting was changed to* $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_2=(-)^{\leftarrow}_L}$ $\mathbf{LoUQuant}$ *in* [22]. *The machinery can be generalized to an arbitrary variety* $\mathbf{A}$ *and the theory* $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{\mathcal{S}^A_{\mathbf{A}}=(-)^{\leftarrow}_A} \mathbf{LoA}$*, which unites the previous items in one common* fixed-basis *framework.*

(4) $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}^{\mathbf{C}}_1=(-)^{\leftarrow}} \mathbf{DmLoc}$*, where* $\mathbf{C}$ *is a subcategory of* $\mathbf{DmLoc}$*, gives the variable-basis poslat approach of S. E. Rodabaugh* [54]. *The setting has been extended to* $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}^{\mathbf{C}}_2=(-)^{\leftarrow}} \mathbf{LoUSQuant}$ *in* [61] *and then reduced to* $\mathbf{Set} \times \mathbf{Loc} \xrightarrow{\mathcal{R}_3=(-)^{\leftarrow}} \mathbf{Loc}$ *in* [11, 14].

(5) $\mathbf{Set} \times \mathbf{FuzLat} \xrightarrow{\mathcal{E}=(-)^{\leftarrow}} \mathbf{FuzLat}$ *provides the variable-basis approach of P. Eklund* [17]*, motivated by those of S. E. Rodabaugh* [54] *and B. Hutton* [34]. *Notice that* $\mathbf{FuzLat}$ *is the dual of the variety* $\mathbf{HUT}$ *of completely distributive DeMorgan frames called* Hutton algebras [57]. *The machinery can be generalized to an arbitrary variety* $\mathbf{A}$ *and the theory* $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{S}^{\mathbf{C}}_{\mathbf{A}}=(-)^{\leftarrow}} \mathbf{LoA}$*, which unites the previous items in one common* variable-basis *framework.*

On the next step, we provide another level of abstraction, which has never been used in the above-mentioned theories of S. E. Rodabaugh.

Definition 6.3. Let $\mathbf{X}$ be a category and let $\mathcal{T} = (P, (\| - \|, \mathbf{B}))$ contain a vbp-theory $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$ in the category $\mathbf{X}$ and a reduct $(\| - \|, \mathbf{B})$ of $\mathbf{A}$. The *variety-based topological theory* (*vt-theory*) *in* $\mathbf{X}$ *induced by* $\mathcal{T}$ is the functor $\mathbf{X} \xrightarrow{T=\|-\|^{op}\circ P} \mathbf{LoB}$.

Since a vt-theory T is completely determined by the respective pair $\mathcal{T}$, we use occasionally the notation $(P, \mathbf{B})$ instead of T. It is important to underline that the aim of an additional level of abstraction is to remove the unused topological structure provided by powerset theories, the move, motivated by the observation that the standard backward powerset theory is based in Boolean algebras, whereas the respective topological theory is reduced to frames (the case of closure spaces mentioned below provides another good example). On the other hand, the case of coincidence between powerset and topological theories is not excluded in our framework. The reader will see that the subsequent developments will often provide a topological theory only, without any explicit reference to its generating powerset theory.

Definition 6.4. Let T be a vt-theory in a category $\mathbf{X}$. $\mathbf{Top}(T)$ is the category, concrete over $\mathbf{X}$, whose objects (*T-topological spaces*) are pairs (X, τ), comprising an $\mathbf{X}$-object X and a subalgebra τ of $T(X)$ (*T-topology* on X), and whose morphisms $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are those $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$, which satisfy $((T(f))^{op})^{\rightarrow}(\sigma) \subseteq \tau$ (*T-continuity*).

The significance of the category $\mathbf{Top}(T)$ is the fact that it unites many of the existing topological frameworks in mathematics. To give the reader the flavor of their abundance and the fruitfulness of the new unifying framework, we provide a short list of examples illustrating the notion of catalg topology.

Example 6.5.

(1) $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ *is isomorphic to the category* $\mathbf{Top}$ *of topological spaces and continuous maps.*

(2) $\mathbf{Top}((\mathcal{P}, \mathbf{CSL}))$ *is isomorphic to the category* $\mathbf{Cls}$ *of closure spaces and continuous maps, studied by D. Aerts* et al. [2, 3].

(3) $\mathbf{Top}((\mathcal{Z}, \mathbf{Frm}))$ *is isomorphic to the category* $\mathbb{I}$-$\mathbf{Top}$ *of fixed-basis fuzzy topological spaces, introduced by C. L. Chang* [8].

(4) $\mathbf{Top}((\mathcal{G}_2, \mathbf{UQuant}))$ *is isomorphic to the category* L-$\mathbf{Top}$ *of fixed-basis* L*-fuzzy topological spaces of J. A. Goguen* [22].

(5) $\mathbf{Top}((\mathcal{R}_i^{\mathbf{C}}, \mathbf{USQuant}))$ *is isomorphic to the category* $\mathbf{C}$-$\mathbf{Top}_i$*,* $i \in \{1, 2\}$ *for variable-basis poslat topology of S. E. Rodabaugh* [54, 61].

(6) $\mathbf{Top}((\mathcal{E}, \mathbf{Frm}))$ *is isomorphic to the category* $\mathbf{FUZZ}$ *for variable-basis poslat topology of P. Eklund* [17]*, motivated by those of S. E. Rodabaugh* [54] *and B. Hutton* [34].

(7) $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{A}, \mathbf{A}))$ *(resp.* $\mathbf{Top}((\mathcal{S}_{\mathbf{A}}^{\mathbf{LoA}}, \mathbf{A}))$*) is isomorphic to the fixed- (resp. variable-) basis category* A-$\mathbf{Top}$ *(resp.* $\mathbf{LoA}$-$\mathbf{Top}$*) used in the former approach to catalg topologies of* [70] *(resp.* [69]*) as well as in the previous sections of this paper (Definition 2.7).*

The reader should notice the fact that the second item of Example 6.5 is never included in the setting of topological theories of S. E. Rodabaugh [61], which are based explicitly on s-quantales (and, therefore, on $\bigvee$-semilattices), whereas closure spaces rely on c-semilattices (and, therefore, on $\bigwedge$-semilattices).

It appears (as an experienced reader might guess) that in order to deal successfully with the categories of the form $\mathbf{Top}(T)$, it is enough to consider their generating topological theories T.

Proposition 6.6. *Let* $\mathbf{X} \xrightarrow{T_1} \mathbf{LoA}$*,* $\mathbf{Y} \xrightarrow{T_2} \mathbf{LoA}$ *be vt-theories, let* $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ *be a functor, and let* $T_2 \circ F \xrightarrow{\eta} T_1$ *be a natural transformation. There exists a functor* $\mathbf{Top}(T_1) \xrightarrow{\mathrm{H}_\eta} \mathbf{Top}(T_2)$ *given by* $\mathrm{H}_\eta((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (F(X_1), (\eta_{X_1}^{op})^{\rightarrow}(\tau_1)) \xrightarrow{F(f)} (F(X_2), (\eta_{X_2}^{op})^{\rightarrow}(\tau_2))$*.*

Proof. It will be enough to show that the functor H_η preserves continuity and that follows immediately from commutativity of the diagram

$$\begin{array}{ccc} T_1(X_2) & \xrightarrow{\eta_{X_2}^{op}} & T_2 \circ F(X_2) \\ {\scriptstyle (T_1(f))^{op}} \downarrow & & \downarrow {\scriptstyle (T_2 \circ F(f))^{op}} \\ T_1(X_1) & \xrightarrow[\eta_{X_1}^{op}]{} & T_2 \circ F(X_1), \end{array}$$

since $((T_2 \circ F(f))^{op})^{\rightarrow}((\eta^{op}_{X_2})^{\rightarrow}(\tau_2)) = ((T_2 \circ F(f))^{op} \circ \eta^{op}_{X_2})^{\rightarrow}(\tau_2) = (\eta^{op}_{X_1} \circ (T_1(f))^{op})^{\rightarrow}(\tau_2) \subseteq (\eta^{op}_{X_1})^{\rightarrow}(\tau_1)$. □

As an example of the obtained result, one can look at Proposition 5.2 and the remark just afterward, providing two functors $\mathbf{Top}((-)^{\leftarrow}_{\Omega G}) \xrightarrow{\mathrm{H}_{\Vdash^{op}}} \mathbf{Top}((-)^{\leftarrow}_{\Sigma G})$ and $\mathbf{Top}((-)^{\leftarrow}_{\Omega F}) \xrightarrow{\mathrm{H}_{\Vdash^{op}}} \mathbf{Top}((-)^{\leftarrow}_{\Sigma F})$, which essentially are the fixed-basis functors G-$\mathbf{Top} \xrightarrow{\mathrm{H}^G_{\mathbf{ATT}}} \Sigma\, G$-$\mathbf{Top}$ and F-$\mathbf{Top} \xrightarrow{\mathrm{H}^F_{\mathbf{Att}}} \Sigma\, F$-$\mathbf{Top}$ of Definitions 2.17 and 3.4 respectively. Moreover, it appears that a more general framework is available. Start by defining the required topological theories, together with an additional functor and a natural transformation.

Proposition 6.7. *There exist topological theories*

(1) $\mathbf{Set} \times \mathbf{LoATTC} \xrightarrow{T^{\mathbf{ATT}}_{\Omega}} \mathbf{LoC}$, *where* $T^{\mathbf{ATT}}_{\Omega}((X_1, G_1) \xrightarrow{(f,g)} (X_2, G_2)) = (\Omega\, G_1)^{X_1} \xrightarrow{((f,(\Omega\, g)^{op})^{\leftarrow})^{op}} (\Omega\, G_2)^{X_2}$*;*

(2) $\mathbf{Set} \times \mathbf{LoC} \xrightarrow{T^{\mathbf{ATT}}_{\Sigma}} \mathbf{LoC} = \mathbf{Set} \times \mathbf{LoC} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoC}$*;*

together with a functor $\mathbf{Set} \times \mathbf{LoATTC} \xrightarrow{\mathrm{K}^{\mathbf{ATT}}_{\times\,\Omega}} \mathbf{Set} \times \mathbf{LoC}$, $\mathrm{K}^{\mathbf{ATT}}_{\times\,\Omega}((X_1, G_1) \xrightarrow{(f,g)} (X_2, G_2)) = (X_1 \times |\,\Omega\, G_1|, \Sigma\, G_1) \xrightarrow{(f \times (\Omega\, g)^{*\,op}, (\Sigma\, g)^{op})} (X_2 \times |\,\Omega\, G_2|, \Sigma\, G_2)$ *and a natural transformation* $T^{\mathbf{ATT}}_{\Sigma} \circ \mathrm{K}^{\mathbf{ATT}}_{\times\,\Omega} \xrightarrow{((-)^{\Vdash})^{op}} T^{\mathbf{ATT}}_{\Omega}$ *given by the maps of Proposition 5.1.*

Proof. Since the definitions of the functors are straightforward, the only thing to verify is correctness of the definition of the natural transformation. Consider a $\mathbf{Set} \times \mathbf{LoATTC}$-morphism $(X_1, G_1) \xrightarrow{(f,g)} (X_2, G_2)$ and check commutativity of the diagram

$$\begin{array}{ccc}
(\Omega\, G_2)^{X_2} & \xrightarrow{(-)^{\Vdash}_{(X_2,G_2)}} & (\Sigma\, G_2)^{X_2 \times |\,\Omega\, G_2|} \\
{\scriptstyle (f,(\Omega\, g)^{op})^{\leftarrow}} \downarrow & & \downarrow {\scriptstyle (f \times (\Omega\, g)^{*\,op}, (\Sigma\, g)^{op})^{\leftarrow}} \\
(\Omega\, G_1)^{X_1} & \xrightarrow[(-)^{\Vdash}_{(X_1,G_1)}]{} & (\Sigma\, G_1)^{X_1 \times |\,\Sigma\, G_1|}.
\end{array}$$

Given $\alpha \in (\Omega\, G_2)^{X_2}$ and $(x, b) \in X_1 \times |\,\Omega\, G_1|$,

$$\begin{aligned}
((-)^{\Vdash}_{(X_1,G_1)} \circ (f, (\Omega\, g)^{op})^{\leftarrow}(\alpha))(x, b) = (\Omega\, g \circ \alpha \circ f)^{\Vdash}(x, b) = \\
(\Vdash_1(\Omega\, g \circ \alpha \circ f(x)))(b) = (\Sigma\, g \circ \Vdash_2(\alpha \circ f(x))((\Omega\, g)^{*op}))(b) = \\
\Sigma\, g \circ \alpha^{\Vdash}(f(x), (\Omega\, g)^{*op}(b)) = (\Sigma\, g \circ \alpha^{\Vdash} \circ (f \times (\Omega\, g)^{*op}))(x, b) = \\
((f \times (\Omega\, g)^{*op}, (\Sigma\, g)^{op})^{\leftarrow} \circ (-)^{\Vdash}_{(X_2,G_2)}(\alpha))(x, b).
\end{aligned}$$

□

Propositions 6.6, 6.7 give a functor $\mathbf{Top}(T_{\Omega}^{\mathbf{ATT}}) \xrightarrow{\mathrm{H}_{\Vdash^{op}}} \mathbf{Top}(T_{\Sigma}^{\mathbf{ATT}})$, which is essentially (up to the change of the notation for the underlying variety) the functor **LoATTC-Top** $\xrightarrow{\mathrm{H}_{\mathbf{ATT}}}$ **LoC-Top** of Definition 2.17. In a similar way, one obtains the functor $\mathbf{Top}(T_{\Omega}^{\mathbf{Att}}) \xrightarrow{\mathrm{H}_{\Vdash^{op}}} \mathbf{Top}(T_{\Sigma}^{\mathbf{Att}})$, which essentially provides the functor **LoATTB-Top** $\xrightarrow{\mathrm{H}_{\mathbf{Att}}}$ **LoB-Top** of Definition 3.4. The reader should notice the significant difference between the ways of obtaining the functors $\mathrm{H}_{\mathbf{ATT}}$, $\mathrm{H}_{\mathbf{Att}}$ by Propositions 6.6, 6.7 and in Definitions 2.17, 3.4, the latter relying on the framework of topological systems, whereas the former being based explicitly on catalg topology. It is up to the reader to decide, which way is more applicable in his/her framework. We would just like to notice that the final results of this section clearly show one of the main advantages of the notion of attachment, i.e., the fact that it provides a way of moving (natural transformation) between two topological theories, resulting in a functor between the categories of the respective topological structures.

7. Categorically-algebraic attachment

The attentive reader will easily notice that although the definition of objects of the category **ATTB** of Definition 2.4 provides a straightforward generalization of the attachment notion of [26], the definition of morphism comes essentially "out of the blue", the only its justification being the existence of the functor **LoATTB-Top** $\xrightarrow{\mathrm{E}_{\mathbf{ATT}}}$ **LoB-TopSys** of Proposition 2.13. The simple reason for the occurrence is the fact that the case of attachment morphisms has never been treated in [25, 26] and, therefore, there is actually nothing to compare with. It is the main goal of this section to provide a more trustworthy justification for the definition of morphisms of the category **ATTB**.

We start with some new functors, which will be used in the subsequent procedures. For the sake of convenience, in what follows, we change the notation (kept until now) for the underlying variety of **ATTB** from **B** to **A**.

Proposition 7.1. *Given a variety* $\mathbf{A}$*, there exists a functor* $\mathbf{Set}^{op}\times\mathbf{A} \xrightarrow{(-)^{\leftharpoonup}} \mathbf{A}$ *defined by the formula* $((X_1,A_1) \xrightarrow{(f,\varphi)} (X_2,A_2))^{\leftharpoonup} = A_1^{X_1} \xrightarrow{(f,\varphi)^{\leftharpoonup}} A_2^{X_2}$ *with* $(f,\varphi)^{\leftharpoonup}(\alpha) = \varphi\circ\alpha\circ f^{op}$*. The new functor satisfies the equality* $\mathbf{Set}^{op}\times\mathbf{A} \xrightarrow{(-)^{\leftharpoonup}} \mathbf{A} = (\mathbf{Set}\times\mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA})^{op}$.

Proof. Correctness of the definition of the functor follows from the last claim (backed by Proposition 2.6), which is a consequence of the fact that given $\alpha \in A_1^{X_1}$, $(f,\varphi)^{\leftharpoonup}(\alpha) = \varphi\circ\alpha\circ f^{op} = (f^{op},\varphi^{op})^{\leftarrow}(\alpha)$. □

To underline the motivating setting of the functor $(-)^{\leftarrow}$, the new one uses a similar notation $(-)^{\leftharpoonup}$. The other functors are collected in the next definition.

Definition 7.2. Every variety $\mathbf{A}$ equipped with a functor $\mathbf{A} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ such that $A^* = |A|$, gives rise to the following three functors:

(1) $\mathbf{A} \times \mathbf{A} \xrightarrow{\Pi_1} \mathbf{A}$, $\Pi_1((A_1, A_1') \xrightarrow{(\varphi,\psi)} (A_2, A_2')) = A_1 \xrightarrow{\varphi} A_2$ (the first projection functor);

(2) $\mathbf{A} \times \mathbf{A} \xrightarrow{\mathrm{K}} \mathbf{Set}^{op} \times \mathbf{A} = \mathbf{A} \times \mathbf{A} \xrightarrow{(-)^* \times 1_{\mathbf{A}}} \mathbf{Set}^{op} \times \mathbf{A}$;

(3) $\mathbf{Set}^{op} \times \mathbf{A} \xrightarrow{\mathrm{P}} \mathbf{A} = \mathbf{Set}^{op} \times \mathbf{A} \xrightarrow{(-)^{\leftharpoonup}} \mathbf{A}$.

The next definition shows a more general approach to attachment, with the notion of morphism coming from the existing framework of comma categories.

Definition 7.3. $\mathbf{ATT}^*\mathbf{A}$ is the full subcategory of the comma category $(\Pi_1 \downarrow \mathrm{P} \circ \mathrm{K})$, whose objects are precisely those $(\Pi_1 \downarrow \mathrm{P} \circ \mathrm{K})$-objects $\Pi_1(A_1, A_2) \xrightarrow{\varphi} \mathrm{P} \circ \mathrm{K}(A_1', A_2')$ for which $(A_1, A_2) = (A_1', A_2')$.

A natural question on the equivalence of Definition 7.3 and Definition 2.4 arises. It is the purpose of the next result to answer it positively.

Proposition 7.4. *The categories* $\mathbf{ATTA}$ *and* $\mathbf{ATT}^*\mathbf{A}$ *are isomorphic.*

Proof. An $\mathbf{ATT}^*\mathbf{A}$-object G is a triple $(\Omega\, G, \Sigma\, G, \Vdash)$, where $(\Omega\, G, \Sigma\, G)$ is the object of the product category $\mathbf{A} \times \mathbf{A}$ and $\Pi_1(\Omega\, G, \Sigma\, G) \xrightarrow{\Vdash} \mathrm{P} \circ \mathrm{K}(\Omega\, G, \Sigma\, G) = \Omega\, G \xrightarrow{\Vdash} (\Sigma\, G)^{|\,\Omega\, G|}$ is an $\mathbf{A}$-homomorphism. An $\mathbf{ATT}^*\mathbf{A}$-morphism $G_1 \xrightarrow{f} G_2$ is a pair of $\mathbf{A}$-homomorphisms $(\Omega\, G_1, \Sigma\, G_2) \xrightarrow{(\Omega\, f, \Sigma\, f)} (\Omega\, G_2, \Sigma\, G_2)$ making the following diagram commute:

$$
\begin{array}{ccc}
\Pi_1(\Omega\, G_1, \Sigma\, G_1) & \xrightarrow{\quad \Vdash_1 \quad} & \mathrm{P} \circ \mathrm{K}(\Omega\, G_1, \Sigma\, G_1) \\
{\scriptstyle \Pi_1(f)} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{P} \circ \mathrm{K}(f)} \\
\Pi_1(\Omega\, G_2, \Sigma\, G_2) & \xrightarrow[\quad \Vdash_2 \quad]{} & \mathrm{P} \circ \mathrm{K}(\Omega\, G_2, \Sigma\, G_2)
\end{array}
$$

that in its turn provides commutativity of the next diagram:

$$
\begin{array}{ccc}
\Omega\, G_1 & \xrightarrow{\quad \Vdash_1 \quad} & (\Sigma\, G_1)^{|\,\Omega\, G_1|} \\
{\scriptstyle \Omega\, f} \big\downarrow & & \big\downarrow {\scriptstyle ((\Omega\, f)^*, \Sigma\, f)^{\leftharpoonup}} \\
\Omega\, G_2 & \xrightarrow[\quad \Vdash_2 \quad]{} & (\Sigma\, G_2)^{|\,\Omega\, G_2|},
\end{array}
$$

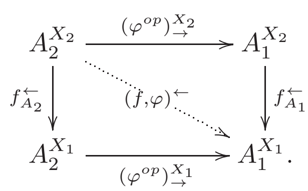
which is equivalent to the fact that given $b_1 \in \Omega\, G_1$ and $b_2 \in \Omega\, G_2$, it follows that $(\Vdash_2 \circ\, \Omega\, f(b_1))(b_2) = ((((\Omega\, f)^*, \Sigma\, f)^{\leftharpoonup} \circ \Vdash_1)(b_1))(b_2) = \Sigma\, f \circ \Vdash_1(b_1) \circ (\Omega\, f)^{*op}(b_2) = (\Sigma\, f \circ \Vdash_1(b_1))((\Omega\, f)^{*op}(b_2))$. Taking together, the remarks provide an isomorphism $\mathbf{ATTA} \xrightarrow{K} \mathbf{ATT}^*\mathbf{A}$, $K(G_1 \xrightarrow{f} G_2) = G_1 \xrightarrow{f} G_2$. □

Having introduced an equivalent definition of attachment, we are going to generalize the results from the end of the last section to the new framework, namely, derive a respective natural transformation between topological theories. Now, however, we would like to provide a more explicit description of the machinery employed. To make the things easier, we begin with several additional properties of powerset operators.

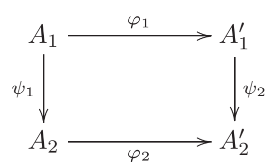
Proposition 7.5. *Given a variety* $\mathbf{A}$ *and a set* X*, there exists a functor* $\mathbf{A} \xrightarrow{(-)^X_{\rightarrow}} \mathbf{A}$, $(A_1 \xrightarrow{\varphi} A_2)^X_{\rightarrow} = A_1^X \xrightarrow{\varphi^X_{\rightarrow}} A_2^X$ *with* $\varphi^X_{\rightarrow}(\alpha) = \varphi \circ \alpha$.

Proof. To show that the functor is correct on morphisms (provides an $\mathbf{A}$-homomorphism), notice that given $\lambda \in \Lambda_{\mathbf{A}}$ and $\alpha_i \in A_1^X$ for $i \in n_\lambda$, it follows that $(\varphi^X_{\rightarrow}(\omega_\lambda^{A_1^X}(\langle \alpha_i \rangle_{n_\lambda})))(x) = \varphi \circ \omega_\lambda^{A_1}(\langle \alpha_i(x) \rangle_{n_\lambda}) = \omega_\lambda^{A_2}(\langle \varphi \circ \alpha_i(x) \rangle_{n_\lambda}) = (\omega_\lambda^{A_2^X}(\langle \varphi \circ \alpha_i \rangle_{n_\lambda}))(x) = (\omega_\lambda^{A_2^X}(\langle \varphi^X_{\rightarrow}(\alpha_i) \rangle_{n_\lambda}))(x)$ for every $x \in X$. □

Notice that the morphism action of the functor of Proposition 7.5 has already been considered in [69, 70], where it has been observed (following, e.g., [56, 59]) that the variable-basis functor of Proposition 2.6 splits up as follows:

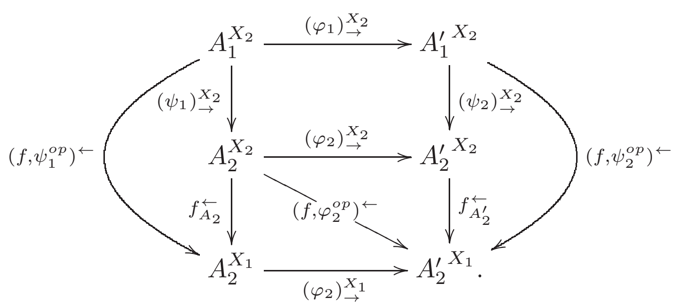


In view of Proposition 7.5 and the above-mentioned remark, every commutative diagram in a variety $\mathbf{A}$



and every map $X_1 \xrightarrow{f} X_2$, provide the following commutative diagram:

(7.1)



The last needed property of powerset operators is in the next proposition.

Proposition 7.6. *For a* **Set** $\times$ **LoA** $\times$ **LoA***-morphism* $(X_1, A_1, B_1) \xrightarrow{(f,\varphi^{op},\psi^{op})} (X_2, A_2, B_2)$*, there exist* **A***-homomorphisms* $(B_i^{|A_i|})^{X_i} \xrightarrow{\Theta_i} B_i^{X_i \times |A_i|}$ *defined by* $(\Theta_i(\alpha))(x,a) = (\alpha(x))(a)$*, making the following diagram commute:*

$$\begin{array}{ccc}
(B_2^{|A_2|})^{X_2} & \xrightarrow{\Theta_2} & B_2^{X_2 \times |A_2|} \\
{\scriptstyle (f,((\varphi^*,\psi)^{\leftarrow})^{op})^{\leftarrow}}\big\downarrow & & \big\downarrow{\scriptstyle (f\times\varphi^{*\,op},\psi^{op})^{\leftarrow}} \\
(B_1^{|A_1|})^{X_1} & \xrightarrow[\Theta_1]{} & B_1^{X_1 \times |A_1|}.
\end{array}$$

Proof. There are two simple challenges to deal with. To show that Θ_i is an **A**-homomorphism, notice that given $\lambda \in \Lambda_{\mathbf{A}}$ and $\alpha_j \in (B_i^{|A_i|})^{X_i}$ for $j \in n_\lambda$, it follows that $(\Theta_i(\omega_\lambda^{(B_i^{|A_i|})^{X_i}}(\langle\alpha_j\rangle_{n_\lambda})))(x,a) = ((\omega_\lambda^{(B_i^{|A_i|})^{X_i}}(\langle\alpha_j\rangle_{n_\lambda}))(x))(a) = (\omega_\lambda^{B_i^{|A_i|}}(\langle\alpha_j(x)\rangle_{n_\lambda}))(a) = \omega_\lambda^{B_i}(\langle(\alpha_j(x))(a)\rangle_{n_\lambda}) = \omega_\lambda^{B_i}(\langle(\Theta_i(\alpha_j))(x,a)\rangle_{n_\lambda}) = (\omega_\lambda^{B_i^{X_i\times|A_i|}}(\langle\Theta_i(\alpha_j)\rangle_{n_\lambda}))(x,a)$ for every $(x,a) \in X_i \times |A_i|$. Commutativity of the above-mentioned diagram follows from the fact that given $\alpha \in (B_2^{|A_2|})^{X_2}$ and $(x_1,a_1) \in X_1 \times |A_1|$, $((\Theta_1 \circ (f,((\varphi^*,\psi)^{\leftarrow})^{op})^{\leftarrow})(\alpha))(x_1,a_1) = (\Theta_1((\varphi^*,\psi)^{\leftarrow} \circ \alpha \circ f))(x_1,a_1) = ((\varphi^*,\psi)^{\leftarrow} \circ \alpha \circ f(x_1))(a_1) = \psi \circ (\alpha \circ f(x_1)) \circ \varphi^{*\,op}(a_1) = \psi \circ (\Theta_2(\alpha))(f(x_1),\varphi^{*\,op}(a_1)) = \psi \circ \Theta_2(\alpha) \circ (f \times \varphi^{*\,op})(x_1,a_1) = ((f \times \varphi^{*\,op},\psi^{op})^{\leftarrow} \circ \Theta_2)(x_1,a_1)$. ☐

Everything in its place, we consider the analogues for our current setting of the functors of Proposition 6.7, which are **Set** $\times$ **LoATT*****A** $\xrightarrow{T_{\Omega}^{\mathbf{ATT}}}$ **LoA**, **Set** $\times$ **LoATT*****A** $\xrightarrow{\mathrm{K}_{\times\,\Omega}^{\mathbf{ATT}}}$ **Set** $\times$ **LoA** and **Set** $\times$ **LoA** $\xrightarrow{T_{\Sigma}^{\mathbf{ATT}}}$ **LoA**. Our aim is to provide an equivalent description of the natural transformation between them given in Proposition 6.7. The new approach is bound to clarify its nature, vaguely touched in Proposition 5.1.

Proposition 7.7. *There is a natural transformation* $T_{\Sigma}^{\mathbf{ATT}} \circ \mathrm{K}_{\times\,\Omega}^{\mathbf{ATT}} \xrightarrow{((-)^{\Vdash})^{op}} T_{\Omega}^{\mathbf{ATT}}$ *defined by* $T_{\Omega}^{\mathbf{ATT}}(X,G) \xrightarrow{(-)^{\Vdash}_{(X,G)}} T_{\Sigma}^{\mathbf{ATT}} \circ \mathrm{K}_{\times\,\Omega}^{\mathbf{ATT}}(X,G) = (\Omega\, G)^X \xrightarrow{\Vdash^X} (\Sigma\, G^{|\,\Omega\, G|})^X \xrightarrow{\Theta} \Sigma\, G^{X\times|\,\Omega\, G|}$*, where the map* Θ *comes from Proposition 7.6.*

Proof. The claim is a consequence of the fact that every **Set** $\times$ **LoATT*****A**-morphism $(X_1, G_1) \xrightarrow{(f,g)} (X_2, G_2)$ makes the following diagram commute:

$$\begin{array}{ccccc}
(\Omega\, G_2)^{X_2} & \xrightarrow{\Vdash_2^{X_2}} & ((\Sigma\, G_2)^{|\,\Omega\, G_2|})^{X_2} & \xrightarrow{\Theta_2} & (\Sigma\, G_2)^{X_2\times|\,\Omega\, G_2|} \\
{\scriptstyle (f,(\Omega\, g)^{op})^{\leftarrow}}\big\downarrow & & \big\downarrow{\scriptstyle (f,(((\Omega\, g)^*,\Sigma\, g)^{\leftarrow})^{op})^{\leftarrow}} & & \big\downarrow{\scriptstyle (f\times(\Omega\, g)^{*\,op},(\Sigma\, g)^{op})^{\leftarrow}} \\
(\Omega\, G_1)^{X_1} & \xrightarrow[\Vdash_1^{X_1}]{} & ((\Sigma\, G_1)^{|\,\Omega\, G_1|})^{X_1} & \xrightarrow[\Theta_1]{} & (\Sigma\, G_1)^{X_1\times|\,\Omega\, G_1|},
\end{array}$$

where the left rectangle uses commutativity of the second diagram of Proposition 7.4 in the form of resulting Diagram (7.1), and the right rectangle is a direct consequence of Proposition 7.6. □

The reader should pay attention to the fact that the new description clarifies the categorical background of the natural transformation in question, which was defined in an algebraic way in Propositions 5.1, 6.7. Categorical approach brings more universality in play, reducing dramatically the algebraic dependence on points, thereby opening a possibility to define attachment in a more general category than **Set**. It will be the subject of our forthcoming papers to provide such an extended definition of attachment.

8. Conclusion: lattice-valued categorically-algebraic topology

The notion of *dual attachment* introduced in this paper clarified completely the categorical nature of the concept of *attachment* considered in [25, 26, 73]. As the main achievement, we showed that it provides a way of interaction (natural transformation) between two topological theories (Propositions 6.7, 7.7), which results in a functor between the respective categories of topological structures (see remarks after Proposition 6.7). It was the framework of *categorically-algebraic* (*catalg*) *topology*, which helped to discover this important property. Moreover, we have already remarked (see, e.g., Example 6.5) that the catalg approach incorporates the majority of the existing (lattice-valued) topological settings, erasing the border between crisp and many-valued frameworks. A significant drawback of the concept, however, is its inability to include the theory of (L, M)-*fuzzy topological spaces* of T. Kubiak and A. Šostak [40]. The striking difference of their approach from those used in the paper is the fact that a topological space is defined as a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a *lattice-valued subalgebra* of $T(X)$ for an appropriate topological theory T of Definition 6.3. This observation in hand, the concept of *lattice-valued catalg topology* has been introduced in [65], based in a suitably defined notion of *lattice-valued algebra*. The latter notion has already appeared in [68], motivated by the concept of *fuzzy group* of A. Rosenfeld [63] and its generalization of J. M. Anthony and H. Sherwood [4]. The employed machinery goes in line with the general procedure of, e.g., J. N. Mordeson and D. S. Malik [47] as follows.

Definition 8.1. Let $\mathbf{A}$, $\mathbf{L}$ be varieties, the latter having the variety $\mathbf{CSLat}(\bigvee)$ as a reduct, and let $\mathbf{C}$ be a subcategory of $\mathbf{L}$. An $(\mathbf{A}, \mathbf{C})$-*algebra* is a triple (A, μ, L), where A is an $\mathbf{A}$-algebra, L is a $\mathbf{C}$-algebra and $|A| \xrightarrow{\mu} |L|$ is a map such that for every $\lambda \in \Lambda$ and every $\langle a_i \rangle_{n_\lambda} \in A^{n_\lambda}$, $\bigwedge_{i \in n_\lambda} \mu(a_i) \leq \mu(\omega_\lambda^A(\langle a_i \rangle_{n_\lambda}))$. An $(\mathbf{A}, \mathbf{C})$-*homomorphism* $(A_1, \mu_1, L_1) \xrightarrow{(\varphi,\psi)} (A_2, \mu_2, L_2)$ is an $\mathbf{A} \times \mathbf{C}$-morphism $(A_1, L_1) \xrightarrow{(\varphi,\psi)} (A_2, L_2)$ such that $\psi \circ \mu_1(a) \leq \mu_2 \circ \varphi(a)$ for every $a \in A_1$. $\mathbf{C}$-$\mathbf{A}$ is the category of $(\mathbf{A}, \mathbf{C})$-algebras and $(\mathbf{A}, \mathbf{C})$-homomorphisms, which is concrete over the product category $\mathbf{A} \times \mathbf{C}$.

An important moment arising from the definition should be underlined at once. In [15], A. Di Nola and G. Gerla introduced the category $\mathcal{C}(\tau)$ as a "general approach to the theory of fuzzy algebras". It is easy to see that $\mathcal{C}(\tau)$ is isomorphic to the subcategory $\mathbf{CLat\text{-}Alg}(\boldsymbol{\Omega})^{=}$ of $\mathbf{CLat\text{-}Alg}(\boldsymbol{\Omega})$, with the same objects and with morphisms $(A_1, \mu_1, L_1) \xrightarrow{(\varphi,\psi)} (A_2, \mu_2, L_2)$ satisfying the identity $\psi \circ \mu_1 = \mu_2 \circ \varphi$ (reflected in the notation "$(-)^{=}$"). Moreover, [15] started to develop the theory of fuzzy universal algebra, some results of which can be easily extended to our approach. Bound to the topological nature of this paper, we will only notice that the category **C-A** provides a more appropriate fuzzification of universal algebra, which fuzzifies not only algebras, but also (and that is more important) their respective homomorphisms.

The related topological stuff is an easy modification of Definitions 6.3, 6.4.

Definition 8.2. Let T be a vt-theory in a category $\mathbf{X}$, let $\mathbf{L}$ be a variety having $\mathbf{CSLat}(\bigvee)$ as a reduct, and let $\mathbb{L}$ be a subcategory of $\mathbf{LoL}$. An *$\mathbb{L}$-valued vt-theory in $\mathbf{X}$ induced by T and $\mathbb{L}$* is the pair $(T, \mathbb{L})$.

Definition 8.3. Let $(T, \mathbb{L})$ be an $\mathbb{L}$-valued vt-theory in a category $\mathbf{X}$. $\mathbb{L}\mathbf{Top}(T)$ is the category, concrete over $\mathbf{X} \times \mathbb{L}$, whose objects (*$\mathbb{L}$-valued T-topological spaces*) are triples $(X, \mathcal{T}, L)$, comprising an $\mathbf{X}$-object X, an $\mathbb{L}$-object L and a $(\mathbf{B}, \mathbf{Lo}\mathbb{L})$-algebra $(T(X), \mathcal{T}, L)$ (*$\mathbb{L}$-valued T-topology* on X), and whose morphisms $(X, \mathcal{T}, L) \xrightarrow{(f,\psi)} (Y, \mathcal{S}, M)$ are those $\mathbf{X} \times \mathbb{L}$-morphisms $(X, L) \xrightarrow{(f,\psi)} (Y, M)$, which satisfy the property of $(T(X), \mathcal{T}, L) \xrightarrow{(T(f),\psi)} (T(Y), \mathcal{S}, M)$ being a $\mathbf{Lo}(\mathbf{Lo}\mathbb{L}\text{-}\mathbf{B})$-morphism (*$\mathbb{L}$-valued T-continuity*).

It appears that lattice-valued catalg topology is truly a universal one, incorporating all (up to the knowledge of the authors) existing topological settings (including the catalg one). The following examples justify the fruitfulness of the new notion (the reader should notice that an $\mathbb{L}$-valued vt-theory $(T, \mathbb{L})$ is occasionally denoted by $(P, \mathbf{B}, \mathbf{Lo}\mathbb{L})$, to underline its building blocks).

Example 8.4.

1. (1) $\mathbb{L}\mathbf{Top}((\mathcal{S}^{\mathbf{S}_L}_{\mathbf{CLat}}, \mathbf{SFrm}, \mathbf{S}^{\mathbf{CDCLat}}_M))$, *where* **CLat** *is the variety of complete lattices and* **CDCLat** *is its subcategory of completely distributive lattices, provides a categorical accommodation of the theory of (L,M)-fuzzy topological spaces of T. Kubiak and A. Šostak* [40].
2. (2) $\mathbb{L}\mathbf{Top}((\mathcal{R}_3, \mathbf{Frm}, \mathbf{Frm}))$ *is isomorphic to the category* $\mathbf{Loc}\text{-}\mathbf{F}^2\mathbf{Top}$ *of (L, M)-fuzzy topological spaces of J. T. Denniston, A. Melton and S. E. Rodabaugh* [11]*, which was introduced as a variable-basis counterpart of the above-mentioned approach of T. Kubiak and A. Šostak.*
3. (3) $\mathbb{L}\mathbf{Top}((\mathcal{P}, \mathbf{Frm}, \mathbf{S}^{\mathbf{DMLoc}}_L))$ *provides the approach of U. Höhle* [30].
4. (4) $\mathbb{L}\mathbf{Top}(T)$ *for* $\mathbb{L} = \mathbf{S}_{\mathbf{2}}^{\mathbf{CSLat}(\bigvee)}$ *is isomorphic to the category* $\mathbf{Top}(T)$ *introduced in Definition 6.4.*

In view of the above-mentioned remarks, it seems natural to consider the notion of attachment in the more general lattice-valued framework, and, therefore, one can postulate the following open problem.

Problem 8.5. *What will be the concept of lattice-valued catalg attachment?*

It will be the topic of our further research to extend the already developed framework to the new setting.

Table of categories

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications, 2009.

[2] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *State property systems and closure spaces: a study of categorical equivalence*, Int. J. Theor. Phys. **38** (1999), no. 1, 359–385.

[3] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *On the amnestic modification of the category of state property systems*, Appl. Categ. Struct. **10** (2002), no. 5, 469–480.

[4] J. M. Anthony and H. Sherwood, *Fuzzy groups redefined*, J. Math. Anal. Appl. **69** (1979), 124–130.

[5] M. Barr, **-Autonomous Categories. With an appendix by Po-Hsiang Chu*, Springer-Verlag, 1979.

[6] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc. **31** (1935), 433–454.

[7] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, 1981.

[8] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[9] P. M. Cohn, *Universal Algebra*, D. Reidel Publ. Comp., 1981.

[10] C. De Mitri and C. Guido, *Some remarks on fuzzy powerset operators*, Fuzzy Sets Syst. **126** (2002), no. 2, 241–251.

[11] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Interweaving algebra and topology: lattice-valued topological systems*, Fuzzy Sets Syst. (Special Issue: Linz 2009), in press.

[12] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued topological systems*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler Universität, Linz, 2009, pp. 24–31.

[13] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued predicate transformers and interchange systems*, Abstracts of the 31st Linz Seminar on Fuzzy Set Theory (P. Cintula, E. P. Klement, and L. N. Stout, eds.), Johannes Kepler Universität, Linz, 2010, pp. 31–40.

[14] J. T. Denniston and S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaest. Math. **32** (2009), no. 2, 139–186.

[15] A. Di Nola and G. Gerla, *Lattice valued algebras*, Stochastica **11** (1987), no. 2-3, 137–150.

[16] E. W. Dijkstra, *A Discipline of Programming*, Prentice-Hall: Englewood, 1976.

[17] P. Eklund, *Categorical Fuzzy Topology*, Ph. D. thesis, Åbo Akademi, 1986.

[18] W. Gähler, *Monadic Topology – A New Concept of Generalized Topology*, Recent Developments of General Topology and its Applications (W. Gähler, ed.), Akademie-Verlag, Berlin, 1992, pp. 136–149.

[19] B. Ganter and R. Wille, *Formale Begriffsanalyse. Mathematische Grundlagen*, Springer, 1996.

[20] G. Gerla, *Representations of fuzzy topologies*, Fuzzy Sets Syst. **11** (1983), 103–113.

[21] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.

[22] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43** (1973), 734–742.

[23] G. Grätzer, *Universal Algebra*, 2nd ed., Springer, 2008.

[24] C. Guido, *Powerset Operators Based Approach to Fuzzy Topologies on Fuzzy Sets*, Topological and Algebraic Structures in Fuzzy Sets (S. E. Rodabaugh and E. P. Klement, eds.), Kluwer Academic Publishers, 2003, pp. 401–413.

[25] C. Guido, *Attachment between fuzzy points and fuzzy sets*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler Universität, Linz, 2009, pp. 52–54.

[26] C. Guido, *Fuzzy points and attachment*, Fuzzy Sets Syst. **161**, no. 16 (2010), 2150–2165.

[27] C. Guido and V. Scarciglia, *L-topological spaces as spaces of points*, Fuzzy Sets Syst. **173**, no. 1 (2011), 45–59.

[28] H. Herrlich and G. E. Strecker, *Category Theory*, 3rd ed., Heldermann Verlag, 2007.

[29] D. Hofmann, *Topological theories and closed objects*, Adv. Math. **215**, no. 2 (2007), 789–824.

[30] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl. **78** (1980), 659–673.

[31] U. Höhle, *Many Valued Topology and its Applications*, Boston, Kluwer Academic Publishers, 2001.

[32] U. Höhle, *A note on the hypergraph functor*, Fuzzy Sets Syst. **131**, no. 3 (2002), 353–356.

[33] U. Höhle and A. P. Šostak, *Axiomatic Foundations of Fixed-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Kluwer Academic Publishers, 1999, pp. 123–272.

[34] B. Hutton, *Products of fuzzy topological spaces*, Topology Appl. **11** (1980), 59–67.

[35] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[36] W. Kotzé and T. Kubiak, *Fuzzy topologies of Scott continuous functions and their relation to the hypergraph functor*, Quaest. Math. **15**, no. 2 (1992), 175–187.

[37] D. Kruml, *Spatial quantales*, Appl. Categ. Struct. **10**, no. 1 (2002), 49–62.

[38] D. Kruml and J. Paseka, *Algebraic and Categorical Aspects of Quantales*, Handbook of Algebra (M. Hazewinkel, ed.), vol. 5, Elsevier, 2008, pp. 323–362.

[39] T. Kubiak, *On Fuzzy Topologies*, Ph.D. thesis, Adam Mickiewicz University, Poznań, Poland, 1985.

[40] T. Kubiak and A. Šostak, *Foundations of the theory of (L,M)-fuzzy topological spaces*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler Universität, Linz, 2009, pp. 70–73.

[41] F. W. Lawvere, *Functorial Semantics of Algebraic Theories*, Ph. D. thesis, Columbia University, 1963.

[42] Y.-M. Liu, *An analysis on fuzzy membership relation in fuzzy set theory*, Fuzzy information, knowledge representation and decision analysis, Proc. IFAC Symp., Marseille 1983, IFAC Proc. Ser. 6, 115-122 (1984).

[43] Y.-M. Liu and M.-K. Luo, *Fuzzy Topology*, World Scientific, 1997.

[44] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.

[45] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer, 1998.

[46] E. G. Manes, *Algebraic Theories*, Springer-Verlag, 1976.

[47] J. N. Mordeson and D. S. Malik, *Fuzzy Commutative Algebra*, World Scientific, 1998.

[48] V. Pratt, *Chu spaces*, Coimbra: Universidade de Coimbra, Departamento de Matemátiqca. Textos Mat., Sér. B. **21** (1999), 39–100.

[49] P.-M. Pu and Y.-M. Liu, *Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571–599.

[50] P.-M. Pu and Y.-M. Liu, *Fuzzy topology II: Product and quotient spaces*, J. Math. Anal. Appl. **77** (1980), 20–37.

[51] A. Pultr and S. E. Rodabaugh, *Category theoretic aspects of chain-valued frames: part I: categorical foundations*, Fuzzy Sets Syst. **159**, no. 5 (2008), 501–528.

[52] A. Pultr and S. E. Rodabaugh, *Category theoretic aspects of chain-valued frames: part II: applications to lattice-valued topology*, Fuzzy Sets Syst. **159**, no. 5 (2008), 529–558.

[53] S. E. Rodabaugh, *The Hausdorff separation axiom for fuzzy topological spaces*, Topology Appl. **11** (1980), 319–334.

[54] S. E. Rodabaugh, *A categorical accommodation of various notions of fuzzy topology*, Fuzzy Sets Syst. **9** (1983), 241–265.

[55] S. E. Rodabaugh, *Point-set lattice-theoretic topology*, Fuzzy Sets Syst. **40**, no. 2 (1991), 297–345.

[56] S. E. Rodabaugh, *Powerset operator based foundation for point-set lattice-theoretic (poslat) fuzzy set theories and topologies*, Quaest. Math. **20**, no. 3 (1997), 463–530.
[57] S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Kluwer Academic Publishers, 1999, pp. 273–388.
[58] S. E. Rodabaugh, *Fuzzy Real Lines and Dual Real Lines as Poslat Topological, Uniform, and Metric Ordered Semirings with Unity*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Kluwer Academic Publishers, 1999, pp. 607–631.
[59] S. E. Rodabaugh, *Powerset Operator Foundations for Poslat Fuzzy Set Theories and Topologies*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Dordrecht: Kluwer Academic Publishers, 1999, pp. 91–116.
[60] S. E. Rodabaugh, *Necessary and sufficient conditions for powersets in* **Set** *and* **Set**×**C** *to form algebraic theories*, Abstracts of the 26th Linz Seminar on Fuzzy Set Theory (S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, eds.), Johannes Kepler Universität, Linz, 2005, pp. 89–97.
[61] S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, Int. J. Math. Math. Sci. **2007** (2007), 1–71.
[62] S. E. Rodabaugh, *Relationship of algebraic theories to powersets over objects in* **Set** *and* **Set**×**C**, Fuzzy Sets Syst. **161**, no. 3 (2010), 453–470.
[63] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
[64] S. Solovjovs, *Categorically-algebraic topology*, Abstracts of the International Conference on Algebras and Lattices (Jardafest), Charles University, Prague, 2010, pp. 20–22.
[65] S. Solovjovs, *Lattice-valued categorically-algebraic topology*, Abstracts of the 91st Peripatetic Seminar on Sheaves and Logic (PSSL 91), University of Amsterdam, Amsterdam, 2010, p. 21.
[66] S. Solovyov, *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets Syst. (Special Issue: Linz 2009), in press.
[67] S. Solovyov, *Categorically-algebraic topology*, submitted to Fuzzy Sets Syst. (Special Issue: Linz 2010).
[68] S. Solovyov, *Fuzzy algebras as a framework for fuzzy topology*, Fuzzy Sets Syst. **173**, no. 1 (2011), 81–99.
[69] S. Solovyov, *Categorical frameworks for variable-basis sobriety and spatiality*, Math. Stud. (Tartu) **4** (2008), 89–103.
[70] S. Solovyov, *Sobriety and spatiality in varieties of algebras*, Fuzzy Sets Syst. **159**, no. 19 (2008), 2567–2585.
[71] S. Solovyov, *Categorically-algebraic dualities*, Acta Univ. M. Belii, Ser. Math. **17** (2010), 57–100.
[72] S. Solovyov, *Categorically-algebraic frameworks for Priestley duality*, Contr. Gen. Alg. **19** (2010), 187–208.
[73] S. Solovyov, *Hypergraph functor and attachment*, Fuzzy Sets Syst. **161**, no. 22 (2010), 2945–2961.
[74] S. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, Soft Comput. **14**, no. 10 (2010), 1059–1068.
[75] S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.
[76] R. Wille, *Restructuring lattice theory: An approach based on hierarchies of concepts*, Ordered sets, Proc. NATO Adv. Study Inst., Banff/Can. 1981, 445-470.
[77] L. A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965), 338–365.

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