Extending maps between pre-uniform spaces

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Abstract

We give sufficient conditions on a uniformly continuous map \( f: (X, U) \to (Y, V) \) between completable \( T_1 \)-pre-uniform spaces \( (X, U) \) and \( (Y, V) \) to have a continuous or a uniformly continuous extension \( \hat{f}: \hat{X} \to \hat{Y} \) between the corresponding completions.

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1. Preliminary results

The basic concepts used in this paper: pre-uniformity bases, Cauchy or minimal filters, round, weakly round or strongly round filters and completion conditions are given in [1]. The concept of pre-uniform basis appeared in 1970 under the name of structure [3]. However, non Hausdorff pre-uniform spaces were very seldom considered in Harris monography.

\( T_1 \)-pre-uniform spaces have an important property: Every Cauchy filter contains a unique weakly round filter and every neighborhood filter is weakly round. The set of weakly round filters \( \hat{X} \) of a \( T_1 \)-pre-uniform space has a complete \( T_1 \)-pre-uniform basis \( \hat{U} \) such that the map \( h: (X, U) \to (\hat{X}, \hat{U}) \) which assigns to each \( x \in X \) its neighborhood filter is a uniform embedding. Hence, any uniformly continuous map \( \varphi: (X, U) \to (Y, V) \) between \( T_1 \)-pre-uniform spaces induces a map \( \hat{\varphi}: (\hat{X}, \hat{U}) \to (\hat{Y}, \hat{V}) \) which sends every weakly round filter \( \xi \in \hat{X} \) into the unique weakly round filter \( N \) in \( \hat{Y} \) which is contained in the Cauchy filter

\[ \varphi(\xi) = \{ \varphi(L) \mid L \in \xi \}^+ \]

(For every subfamily \( G \) of the power set of a set \( Z \), we define \( G^+ = \{ L \subseteq Z \mid \text{for some } G \in G, G \subseteq L \} \)). If \( k: (Y, V) \to (\hat{Y}, \hat{V}) \) is the canonical uniform
embedding, i.e. \( k(y) = \) neighborhood filter of \( y \), we have the relation \( \hat{\varphi} \circ h = \hat{k} \circ \varphi \). In this paper, we find conditions on \( \varphi, U \) and \( V \) which insure that \( \hat{\varphi} \) is continuous or uniformly continuous.

2. Main results

We start this section with a lemma.

Lemma 2.1. Let \( (X, U) \) be a \( T_1 \)-pre-uniform space and suppose \( (X, \tau_u) \) is a \( T_1 \)-space. Then every Cauchy filter \( \xi \) in \( (X, U) \) contains a unique minimal filter \( \xi' \).

Proof. We know \( \xi' = \{ S_T^{**}(\xi, \alpha) \mid \alpha \in U \} \) is \( U \)-minimal and is contained in \( \xi \), where

\[
S_T^{**}(\xi, \alpha) = \bigcup \{ L \mid L \in \alpha \cap \xi \}.
\]

Suppose \( \mathcal{N} \subseteq \xi \) is another \( U \)-minimal filter.

Therefore, \( \mathcal{N}' = \mathcal{N} \subseteq \xi' \). The minimal property of \( \xi' \) implies that \( \mathcal{N}' = \mathcal{N} = \xi' \). \( \square \)

We give two cases in which \( \hat{\varphi} \) is uniformly continuous.

Lemma 2.2. Suppose for each \( \xi \in \tilde{X} - h(X), \varphi(\xi) = \varphi(\xi)' \). Then \( \hat{\varphi} \) is uniformly continuous.

Proof. Let \( \beta \in V \) and let \( \alpha \in U \) be such that \( \alpha \leq \varphi^{-1}(\beta) \). We shall prove that \( \hat{\alpha} \leq \hat{\varphi}^{-1}(\hat{\beta}) \). Let \( A \in \alpha \) and \( B \in \beta \) be such that \( A \subseteq \varphi^{-1}(B) \). We claim that \( \hat{A} \subseteq \hat{\varphi}^{-1}(\hat{B}) \). Let us take \( \xi \in \hat{A} \). Then \( \hat{A} \subseteq \hat{\varphi}^{-1}(\hat{B}) \). We claim that \( \hat{A} \subseteq \hat{\varphi}^{-1}(\hat{B}) \). Let us take \( \xi \in \hat{\varphi}(\xi)' \). Then \( \hat{A} \subseteq \hat{\varphi}^{-1}(\hat{\varphi}(\xi)) \). Since \( \varphi(A) \subseteq B \), we have also \( B \in \varphi(\xi) \). Therefore, \( \hat{\varphi}(\xi) = \varphi(\xi) \in \hat{B} \) and the proof is complete. \( \square \)

Lemma 2.3. If \( (Y, V) \) is a semi-uniform space, \( \hat{\varphi} \) is uniformly continuous.

Proof. Let \( \beta \in V \). Since \( (Y, V) \) is a semi-uniform space, there exists a cover \( \gamma \in V \) which satisfies the following condition:

**Su** For each \( C \in \gamma \), there exists \( \delta_C \in V \) and \( B_C \in \beta \) such that \( S_T(C, \delta_C) \subseteq B_C \).

Let \( \alpha \in U \) be such that \( \alpha \leq \varphi^{-1}(\gamma) \). We shall prove that \( \hat{\alpha} \leq \hat{\varphi}^{-1}(\hat{\beta}) \). If \( A \in \alpha \), there exists a set \( C \in \gamma \) such that \( A \subseteq \varphi^{-1}(C) \). By condition **Su**, there exist \( \delta_C \in V \) and \( B_C \in \beta \) such that \( S_T(C, \delta_C) \subseteq B_C \). We claim that \( \hat{A} \subseteq \hat{\varphi}^{-1}(\hat{B}_C) \). If \( \xi \in \hat{A} \), we have \( A \in \xi \). Since \( \varphi(A) \subseteq C \), we have \( C \in \varphi(\xi) \). Therefore, \( S_T(C, \delta_C) \subseteq \varphi(\xi) \). Since \( S_T(C, \delta_C) \subseteq B_C \), we conclude that \( B_C \in \varphi(\xi) \) and \( \hat{\varphi}(\xi) \in \hat{B}_C \). \( \square \)

Lemma 2.4. Let \( X, Y \) be \( T_2 \)-spaces and let \( U, V \), respectively, be the families of densely finite covers of \( X, Y \). Let \( \varphi: X \to Y \) be continuous, open and surjective. Then \( \varphi \) is uniformly continuous as a map from \( (X, U) \) onto \( (Y, V) \).
Lemma 2.6. A non-adherent filter $\mathcal{T}$ in $(X,U)$ is $U$-round if and only if $\mathcal{T}$ has as a basis an ultrafilter of open sets.

Proof. Suppose $\mathcal{T}$ is a non-adherent round filter in $(X,U)$. Let $\mathcal{G}$ be the family of open sets in $\mathcal{T}$ and take an open set $V$ such that $V \cap G \neq \emptyset$ for every $G \in \mathcal{G}$. We have to prove that $V \in \mathcal{T}$ and that will convert $\mathcal{G}$ into an ultrafilter of open sets.

Since $\mathcal{T}$ is non-adherent, the family $\{X - F^- | F \in \mathcal{T}\}$ is an open cover of $X$. Hence, $\alpha = \{V, X - V^- \} \cup \{X - F^- | F \in \mathcal{T}\}$ is a densely finite cover of $X$. Since $\mathcal{T}$ is $U$-Cauchy, we have $V \in \mathcal{T}$ or $X - V^- \in \mathcal{T}$. If we had $X - V^- \in \mathcal{T}$, we use the roundness of $\mathcal{T}$ and find a cover $\beta \in U$ such that $X - V^- \supseteq S_{\mathcal{T}}^*(\mathcal{T}, \beta) = \cup \{B \in \beta | B \cap F \neq \emptyset \}$ for every $F \in \mathcal{T}$). If $G \in \beta \cap \mathcal{T}$, we have $G \subseteq X - V^-$ and hence $V \cap G = \emptyset$, a contradiction. Therefore we must have $V \in \mathcal{T}$ and $\mathcal{G}$ is an ultrafilter of open sets.
Conversely, suppose $\mathcal{G}$ is an ultrafilter of open sets. We have to prove that $\mathcal{T}$ is $\mathcal{U}$-round. We prove first that $\mathcal{T}$ is $\mathcal{U}$-Cauchy. Let $\alpha \in \mathcal{U}$. If $\mathcal{T} \cap \alpha = \emptyset$, then $A \notin \mathcal{T} \cap \tau$ for every $A \in \alpha$. Let $\{A_1,A_2,\ldots,A_n\} \subseteq \alpha$ be such that $X = A_1^- \cup A_2^- \cup \cdots \cup A_n^-$. Since $A_i \notin \mathcal{T} \cap \tau$ and $\mathcal{T} \cap \tau$ is an ultrafilter of open sets, we can find elements $G_i \in \mathcal{T} \cap \tau$ such that $A_i \cap G_i = \emptyset$ ($i = 1,2,\ldots,n$). Hence $(G_1 \cap G_2 \cap \cdots \cap G_n) \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = \emptyset$. But $A_1 \cup A_2 \cup \cdots \cup A_n$ is dense in $X$. Hence $G_1 \cap G_2 \cap \cdots \cap G_n = \emptyset$, a contradiction. We finally prove that $\mathcal{T}$ is $\mathcal{U}$-round. Pick any element $F_0 \in \mathcal{T}$ and consider the cover $\alpha = \{F_0\} \cup \{X - F^- \mid F \in \mathcal{T}\}$. Clearly $S_T^*(\mathcal{T},\alpha) = F_0$ and hence $\mathcal{T}$ is $\mathcal{U}$-round.

In [4] it is proved that every Cauchy filter in $(X,\mathcal{U})$, where $\mathcal{U}$ is the family of densely finite covers of the Hausdorff space $(X,\tau)$, contains an $\mathcal{U}$-round filter and by [1], $(X,\mathcal{U})$ has a completion $(\hat{X},\hat{\mathcal{U}})$ where every $\hat{\mathcal{U}}$-round filter is convergent and the topology $\tau_{\hat{\mathcal{U}}}$ is Hausdorff closed. Besides the completion $(\hat{X},\hat{\mathcal{U}})$, $(X,\tau)$ has the Katetov extension $kX$, which is also Hausdorff closed. In this volume we prove that in general, the extensions $\hat{X}$ and $kX$ are non-equivalent.

3. Applications

**Proposition 3.1.** Let $X$ be a separable, metrizable, dense in itself, 0-dimensional space and let $Z$ be a compact, Hausdorff, separable space. Then there exists a surjective continuous map $g : \hat{X} \to Z$, where $\hat{X}$ is the completion of the pre-uniformity basis of $X$ consisting of all densely finite covers of $X$.

**Proof.** The hypothesis imply the existence of mutually disjoint non-empty open sets $L_1,L_2,\ldots$ such that $X = \bigcup_{n=1}^{\infty} L_n$. The map $\varphi : X \to \mathbb{N}$ where $L_n = \varphi^{-1}(n)$ for each $n \in \mathbb{N}$, is continuous, open and surjective. By 2.4, there exists a continuous surjective extension $\hat{\varphi} : \hat{X} \to \hat{\mathbb{N}}$. But $\hat{\mathbb{N}}$ coincides with the Stone-Čech compactification $\beta \mathbb{N}$ of $\mathbb{N}$ (because a cover $\alpha$ of $\mathbb{N}$ is densely finite if and only if it is finite). On the other hand, by the universal property of $\beta \mathbb{N}$, there exists a continuous surjective map $\psi : \beta \mathbb{N} \to Z$. Hence, $g = \psi \circ \hat{\varphi}$ is a continuous surjective map from $\hat{X}$ onto $Z$. \hfill $\Box$

**Proposition 3.2.** Let $X$ be a non-empty completely metrizable separable space. Then there exists a continuous surjective map $\psi : (\mathbb{N}^\omega)^\rightarrow \to \hat{X}$.

**Proof.** $\mathbb{N}^\omega$ may be identified with the set of irrationals numbers and this space satisfies the conditions of (3.1). On the other hand, there exists a continuous open surjective map $\varphi : \mathbb{N}^\omega \to X$ (see 5.15 in [2]). Using 2.4, we complete the proof. \hfill $\Box$

**Corollary 3.3.** If $Z$ is a Tychonoff separable space which is either compact or completely metrizable, then there exists a continuous surjective map $\psi : (\mathbb{N}^\omega)^\rightarrow \to \hat{Z}$. 

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We finish this paper with a problem:

\textit{Problem 3.4.} Is every Čech-complete separable space a continuous image of \((\mathbb{N}^\omega)\) ?

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