Epimorphisms and maximal covers in categories of compact spaces

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ABSTRACT

The category $C$ is "projective complete" if each object has a projective cover (which is then a maximal cover). This property inherits from $C$ to an epireflective full subcategory $R$ provided the epimorphisms in $R$ are also epi in $C$. When this condition fails, there still may be some maximal covers in $R$. The main point of this paper is illustration of this in compact Hausdorff spaces with a class of examples, each providing quite strange epimorphisms and maximal covers. These examples are then dualized to a category of algebras providing likewise strange monics and maximal essential extensions.

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1. INTRODUCTION

In a category, an essential extension of an object $A$ is a monomorphism $A \xrightarrow{m} B$ for which $km$ monic implies $k$ monic. In recent work [3], the authors have considered the inheritance from a category $C$ to a monocoreflective subcategory $V$ of the property that each object has a unique maximal essential extension. The hypothesis "each monic in $V$ is also monic in $C" was crucial. (The property was deployed to similar ends in [9].) This paper is largely directed at exhibiting in a concrete setting some pathology which can occur in the absence of these hypotheses.

But we shall operate "in dual", as we now describe briefly, and sketch a return to essential extensions in the final §5.
In a category, a cover of the object $X$ is an epimorphism $Y \xrightarrow{g} X$ for which $gf$ epi implies that $f$ is epi. (This definition is dual to "essential extension"). Any projective cover is also a unique maximal cover (2.3). But there are categories with no projectives, and still every object has a unique maximal cover ([3], in dual.)

In compact Hausdorff spaces, Comp, epis are onto and every object has a projective cover (the Gleason cover). For an epireflective subcategory $\mathcal{R}$ of Comp, $\mathcal{R}$ has a non-void projective if and only if epis in $\mathcal{R}$ are onto (3.5) and then the projective covers from Comp are projective covers in $\mathcal{R}$ (3.2).

We begin with a necessary discussion of simple categorical preliminaries, proceed to Comp and two specific epireflective subcategories, then extract what little can be said for an epireflective $\mathcal{R}$ in general. Penultimately, we consider a strongly rigid $E \in \text{Comp}$ and the epireflective subcategory $\mathcal{R}(E)$ which $E$ generates. There are epis not onto, and any nonconstant $E \rightarrow \{0,1\}$ is a maximal cover. Finally, we sketch the dualization of this to a category of algebras, in which any proper $C(E) \rightarrow \mathbb{R}^2$ is a maximal essential extension.

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2. Preliminaries

The context for 2.1 - 2.7 is a fixed category with no hypotheses at all before 2.4. In the following, $g,h,k,\ldots$ are assumed to be morphisms. The terms "morphism" and "map" will be interchangeable.

Definition 2.1.

(a) A morphism $g$ is an epimorphism (epi) if $hg = kg$ implies $h = k$.

(b) The map $g$ is "covering" if epi, and $gf$ epi implies $f$ epi. (Such $g$ could also be called essential epi (or perhaps co-essential epi).) A cover of object $X$ is a pair $(X, g)$ with $Y \xrightarrow{g} X$ covering. Covers of $X$, $(Y, g)$ and $(Y', g')$ are equivalent if there is an isomorphism $h$ with $g'h = g$.

(c) Object $Y$ is cover-complete if, $(Z, k)$ a cover of $Y$ implies $k$ is an isomorphism. A maximal cover of $X$ is a cover $(Y, g)$ with $Y$ cover complete. A unique maximal cover of $X$ is a maximal cover which is equivalent to any other maximal cover of $X$.

(d) Object $P$ is projective if whenever $X \xrightarrow{h} P$ and $X \xrightarrow{g} Y$ is an epi, then there is $Y \xrightarrow{f} P$ with $gf = h$. A projective cover is a cover $(P, p)$ with $P$ projective.

(e) The category is called projectively complete if every object has a projective cover, and (weaker) is said to have enough projectives if for each object $X$ there is $X \xrightarrow{f} P$, $f$ epi and $P$ projective.

The following two elementary propositions are, except for 2.2 (d) and perhaps 2.3 (b), proved (in dual) in [1], 9.14, 9.19, 9.20.
Proposition 2.2.
(a) An isomorphism is covering.
(b) The composition of two covering maps is covering.
(c) If \( g \) and \( gf \) are covering, then \( f \) is covering.
(d) If \( gf \) is covering and \( f \) is epi, then \( f \) is covering.

Proof.
(d) Given such \( gf \) and \( f \), suppose \( fh \) is epi. Note that \( g \) is epi (because \( gf \) is). So, \( g(fh) \) is epi, and \( g(fh) = (gf)h \) shows \( h \) is epi, since \( gf \) is covering.

□

Proposition 2.3.
(a) A projective object is cover-complete.
(b) A projective cover is a unique maximal cover.

Proof.
(b) Suppose \((P, p)\) is a projective cover of \( X \). It is a maximal cover by (a). If \((Y, g)\) is another cover of \( X \), there is \( k \) with \( gk = p \) (since \( P \) is projective and \( g \) is epi). By 2.2 (c), \( k \) is covering, thus an isomorphism if \( Y \) is cover-complete.

□

To proceed further, we require assumptions.

Two Hypotheses 2.4. (to be invoked selectively). Let \( C \) be a category, and \( \mathcal{R} \) a subcategory (always assumed full and isomorphism-closed).

The first condition is on \( C \) alone, and is “the other face” of 2.2 (d):

\((F^o)\) If \( gf \) is covering and \( f \) is epi, then \( g \) is covering

The second condition is on \( \mathcal{R} \subseteq C \), and is the (frequently invalid) converse to the obvious truth “Any \( C \)-epi between \( \mathcal{R} \)-objects is \( \mathcal{R} \)-epi”:

\((S^o)\) Any \( \mathcal{R} \)-epi is \( C \)-epi.

The point of this paper is, in the presence of \((F^o)\), what happens when \((S^o)\) holds (2.7 and §3), and especially what can happen when it fails (§5, §6).

Proposition 2.5. If \( C \) has enough projectives (in particular if \( C \) is projectively complete), then \( C \) satisfies \((F^o)\).

Proof. Consider \( X \xrightarrow{g} Y \xrightarrow{f} Z \) with \( gf \) covering and \( f \) epi. Since \( gf \) is epi, so is \( g \). Suppose \( Y \xrightarrow{t} T \) has \( gt \) epi; we want \( t \) epi. Take \( T \xrightarrow{e} P \) epi with \( P \) projective. There is \( Z \xrightarrow{h} P \) with \( fh = te \) (since \( f \) is epi). Then, \((gf)h = g(fh) = g(te) = (gt)e \). The last term is epi, and so also the first term. Thus \( h \) is epi (since \( gf \) is covering), and so also \( fh \). Since \( fh = te \), \( t \) is epi. □
Proposition 2.6. Suppose \((S^\circ)\). If \(X, Y \in \mathcal{R}\), and \(X \xrightarrow{g} Y\) is \(\mathcal{C}\)-covering, then \(g\) is \(\mathcal{R}\)-covering.

Proof. Suppose given \(X \xrightarrow{g} Y\) as stated, and \(Y \xrightarrow{f} Z\) with \(Z \in \mathcal{R}\) and \(gf\) \(\mathcal{R}\)-epi. Then \(gf\) is \(\mathcal{C}\)-epi (by \((S^\circ)\)), so \(f\) is \(\mathcal{C}\)-epi (since \(g\) is \(\mathcal{C}\)-covering), thus also \(\mathcal{R}\)-epi (as desired).

We say \((\mathcal{R}, r)\) is epireflective in \(\mathcal{C}\) if \(\mathcal{R}\) is a subcategory of \(\mathcal{C}\), and for each \(Y \in \mathcal{C}\) there is \(rY \in \mathcal{R}\) (the reflection) and \(\mathcal{R}\)-epi \(fr\) \(\mathcal{C}\)-epi (by \((S^\circ)\)), so \(f\) is \(\mathcal{C}\)-epi (since \(g\) is \(\mathcal{C}\)-covering), thus also \(\mathcal{R}\)-epi (as desired).

Proposition 2.7. Suppose that \((\mathcal{R}, r)\) is epireflective in \(\mathcal{C}\), and satisfies \((S^\circ)\).

(a) If \(P\) is projective in \(\mathcal{C}\), then \(rP\) is projective in \(\mathcal{R}\).
(b) Suppose further that \(\mathcal{C}\) satisfies \((F^\circ)\). If \(X \in \mathcal{R}\), and \((P, p)\) is a projective cover in \(\mathcal{C}\) of \(X\), then \((rP, \bar{p})\) is a projective cover in \(\mathcal{R}\) of \(X\).
(c) If \(\mathcal{C}\) is projectively complete, then so is \(\mathcal{R}\) (with projective covers as in \((b)\)).

Proof. (c) (from \((b)\)). 2.5 says \(\mathcal{C}\) satisfies \((F^\circ)\), so \((b)\) applies.

(a) Suppose given \(\mathcal{R}\)-epi \(X \xrightarrow{g} Y\) and any \(X \xrightarrow{f} rP\). By \((S^\circ)\), \(g\) is \(\mathcal{C}\)-epi, so there is \(f_1\) with \(gf_1 = frP\) (since \(P\) is \(\mathcal{C}\)-projective). Next, there is \(f_2\) with \(f_2rP = f_1\), and we have \(frP = gf_1 = g(f_2rP) = (gf_2)rP\). Since \(rP\) is \(\mathcal{C}\)-epi, \(f = gf_2\).

(b) By \((a)\), \(rP\) is \(\mathcal{R}\)-projective. We need that the \(\bar{p}\) in \(\bar{p}rP = p\) is \(\mathcal{R}\)-covering. Since \(rP\) is epi, \((F^\circ)\) says that \(\bar{p}\) is \(\mathcal{C}\)-covering, and thus \(\mathcal{R}\)-covering by 2.5.

Remark 2.8.

(a) [3], 1.2 shows (in dual) that if \(\mathcal{C}\) has unique maximal covers, so does epireflective \(\mathcal{R}\), assuming the conditions \((S^\circ)\) and \((F^\circ)\). The proofs above of 2.7 \((a)\) and \((b)\) are simplified versions of those in [3]. For 2.7 \((c)\), the present 2.7 (new here) allows suppression of the hypothesis \((F^\circ)\).

(b) If in 2.7, \(\mathcal{R}\) already contains every \(\mathcal{C}\)-projective, then 2.7 \((a)\) and \((b)\) simplify in the obvious way. This is the case for \(\mathcal{C} = \text{Comp}\), with \(\mathcal{R}\) having \((S^\circ)\); see 3.2 below.

3. Compact Hausdorff Spaces

\(\text{Comp}\) is the category of compact Hausdorff spaces with continuous functions as maps. A map \(X \xrightarrow{f} Y\) in \(\text{Comp}\) is called irreducible if \(f(Y) = X\), but when \(F \subseteq Y\) (\(F\) closed), \(f(F) \neq X\). The following is mostly due to Gleason.
Proposition 3.1. In $\text{Comp}$:

(a) Epis are onto. (See comment after 3.3 below.)
(b) A map is covering iff it is irreducible.
(c) A space is projective iff it is extremally disconnected (every open set has open closure).
(d) Any object $X$ has a projective cover $(PX, p_X)$; $\text{Comp}$ is projectively complete.
(e) $(F^e)$ holds.

The notation $(P X, p_X)$ is reserved for the rest of the paper; this will always denote the projective cover in $\text{Comp}$ of $X \in \text{Comp}$. Also, for brevity, we shall let $\text{ED}$ stand for the class of extremally disconnected spaces in $\text{Comp}$.

(Considerable literature developed from Gleason’s [6], with various new proofs, generalizations, and variants of the theory. See [2], [8], [14] and their bibliographies.)

Now consider a subcategory $A$ of $\text{Comp}$ (which can be identified with its object class). The family of all subobjects (resp., products) of spaces in $A$ is denoted $S_A$ (resp., $P_A$). (Note that subobjects are closed subspaces.) Kennison [13] has shown that $R$ is epireflective in $\text{Comp}$ iff $R$ is neither $\emptyset$ nor $\{\emptyset\}$ and $R = SP_R$. For $\emptyset \neq X \in \text{Comp}$, let $R(X) = SP\{X\}$; this is the smallest epireflective subcategory containing $X$.

Let $\{0\}$ (resp., $\{0, 1\}$) denote the space with one (resp. two) points. The smallest epireflective is $R(\{0\}) = \{\emptyset, \{0\}\}$; here, $\{0\} \rightarrow \emptyset$ is epi, so epis are not onto. We comment further on this shortly. The next largest is $R(\{0, 1\})$: if $R$ is epireflective and not $R(\{0\})$, there is $X \in R$ with $|X| \geq 2$, thus $\{0, 1\} \in R$, so $R(\{0, 1\}) \subset R$. Note that $R(\{0, 1\}) = \text{Comp}_0$, the class of compact zero-dimensional spaces [5], and $\text{ED} \subset \text{Comp}_0$ [7]. Thus, if $R$ is epireflective and not $R(\{0\})$, $\text{ED} \subset R$.

Corollary 3.2. Suppose $R$ is epireflective and $R$-epis are onto (i.e., $R \subset \text{Comp}$ satisfies $(S^e)$). Then $R$ is projectively complete. In fact, for any $X \in R$, the $R$-(projective cover) is $(PX, p_X)$.

Proof. Apply 3.1, 2.4, and the discussion above. \qed

Proposition 3.3. $\text{Comp}_0$-epis are onto. 3.2 applies to $\text{Comp}_0$.

Proof. The following takes place in $\text{Comp}_0$.

The only $\emptyset \rightarrow Y$ has $Y = \emptyset$ and the map is the identity, which is epi, and technically onto. If $X \neq \emptyset$ then $X \rightarrow \emptyset$ is not epi (since there are different $\{0, 1\} \xrightarrow{h} X$).

Suppose $X \neq \emptyset$, and $X \xrightarrow{g} Y$ is epi. Were $g$ not onto, there would be $p \in X - g(Y)$, and clopen $U$ with $p \notin U \supseteq g(Y)$. Then $h$ constantly 1 and $k$ the characteristic function of $U$ has $h \neq k$ but $hg = kg$. \qed
(To show Comp-epis are onto, argue similarly using $[0, 1]$ instead of $\{0, 1\}$, and using complete regularity of $X$ (i.e. the Tietze-Urysohn Theorem).)

**Remark 3.4.** We do not know if there is epireflective $\mathcal{R}$ different from $\text{Comp}_0$ and $\text{Comp}$, for which epis are onto.

The following (closely related to [3], 4.1) shows that, failing "epis are onto", there are no $\neq \emptyset$ projectives. But there still may be some maximal covers, of at least two sorts, as the examples in §5 show.

**Proposition 3.5.** Suppose (only) $\{0\} \in \mathcal{R}$. The following statements in $\mathcal{R}$ are equivalent.

(a) Epis are onto.
(b) $\{0\}$ is projective.
(c) There is a non-void projective.

**Proof.** (b) $\Rightarrow$ (c) obviously, and (c) $\Rightarrow$ (b) because $\{0\}$ is a retract of any $X \neq \emptyset$, and a retract of a projective is projective.

(a) $\Rightarrow$ (b) because $\{0\}$ is projective in $\text{Comp}$, and if (a) holds, projective in $\mathcal{R}$.

(b) $\Rightarrow$ (a). If $X \xleftarrow{g} Y$ is an epi which is not onto, then there is $p \in X - g(Y)$, and for $X \xleftarrow{h} \{0\}$ defined as $h(0) = p$, there can be no $Y \xleftarrow{f} \{0\}$ with $gf = h$. \qed

Finally, we clarify the situation for $\emptyset$ and for $\mathcal{R}(\{0\})$.

Note the following for any $\mathcal{R} \subseteq \text{Comp}$ with $\emptyset \in \mathcal{R}$.

(i) $\emptyset$ is the initial object of $\mathcal{R}$, i.e., for any $X \in \mathcal{R}$, there is unique $X \leftarrow \emptyset$ (namely, the empty map).

(ii) $\emptyset$ is projective in $\mathcal{R}$.

(iii) If $X \leftarrow \emptyset$ is epi in $\mathcal{R}$, then this is a projective cover.

**Proposition 3.6.** Suppose (only) $\{0\} \in \mathcal{R} = \mathcal{R}$. The following statements in $\mathcal{R}$ are equivalent.

(a) $\mathcal{R} = \mathcal{R}(\{0\})$
(b) $\{0\} \rightarrowtail \emptyset$ is epi (and thus a projective cover).
(c) For any $X \in \mathcal{R}$, $X \rightarrowtail \emptyset$ is epi (and thus a projective cover).

**Proof.** The parenthetical remarks follow from the comments above.

(a) $\Rightarrow$ (c): $\emptyset \rightarrowtail \emptyset$ is epi, and $\{0\} \rightarrowtail \emptyset$ also, since the only map out of $\{0\}$ is the identity.

(c) $\Rightarrow$ (b): since $\{0\} \in \mathcal{R}$.

(b) $\Rightarrow$ (a): If $\mathcal{R} \neq \mathcal{R}(\{0\})$, then there is $X \in \mathcal{R}$ with $|X| \geq 2$, so $\{0, 1\} \in \mathcal{R}$ (since $\mathcal{R} = \mathcal{R}$). Then there are different $\{0, 1\} \xleftarrow{h} \{0\}$ which compose equally with $\{0\} \xrightarrow{k} \emptyset$, so the latter is not epi. \qed
Corollary 3.7. $\mathcal{R}(\{0\})$ is projectively complete, with epis not onto, and is the only epireflective subcategory with these two properties.

Proof. 3.6, (a) $\Rightarrow$ (c) yields the first statement. If epireflective $\mathcal{R}$ has epis not onto, then by 3.5, the only projective is $\emptyset$. If $\mathcal{R}$ is projectively complete, then the projective cover must be $X \twoheadrightarrow \emptyset$. So these are epi, and 3.6 (c) $\Rightarrow$ (a) says $\mathcal{R} = \mathcal{R}(\{0\})$. □

4. When epis may not be onto

Consider $\mathcal{R} \subset \text{Comp}$. We localize the condition "$\mathcal{R}$-epis are onto". Keep in mind that $\mathcal{R}$ might have no projectives (but any $Y \in \mathcal{R} \cap \text{ED}$ is still Comp-projective).

Definition 4.1. "$X$ has $e(\mathcal{R})$" means $X \in \mathcal{R}$, and whenever $X \twoheadrightarrow Y$ is epi in $\mathcal{R}$, then $g$ is onto.

Proposition 4.2. Suppose $\mathcal{R} = S\mathcal{R}$.
(a) If $X \in \mathcal{R} \cap \text{Comp}_{\text{po}}$, then $X$ has $e(\mathcal{R})$.
(b) If $Y \in \mathcal{R} \cap \text{ED}$, then $Y$ is cover-complete in $\mathcal{R}$.

Proof.
(a) Identical to the proof of 3.3.
(b) If $(Z, g)$ is an $\mathcal{R}$-cover of $Y$, then $g$ is onto by (a), so there is $f$ with $gf = \text{id}_Y$, since $Y$ is Comp-projective, and $f \in \mathcal{R}$ (since $Y, Z \in \mathcal{R}$). So $f$ is an $\mathcal{R}$-section. Also, by 2.2, $f$ is an $\mathcal{R}$-covering map, thus $\mathcal{R}$-epi. So $f$ is an $\mathcal{R}$-isomorphism, so is $g$, and therefore $g$ is a Comp-isomorphism, thus a homeomorphism.

□

(The converse to 4.2 (a) fails, with $\mathcal{R} = \text{Comp}$. But see 4.5 below.)

Proposition 4.3. Suppose $X$ has $e(\mathcal{R})$.
(a) If $Y \in \mathcal{R}$ and $X \twoheadrightarrow Y$ is irreducible, then $(Y, g)$ is an $\mathcal{R}$-cover of $X$.
(b) If also $PX \in \mathcal{R}$, and supposing $\mathcal{R} = S\mathcal{R}$, then $(PX, p_X)$ is the unique maximal $\mathcal{R}$-cover of $X$.

Proof. (a) As in the proof of 2.6, mutatis mutandis
(b) By (a), $(PX, p_X)$ is an $\mathcal{R}$-cover, and $PX$ is cover-complete. If $(Y, g)$ is another $\mathcal{R}$-cover of $X$, then $g$ is onto (by $e(\mathcal{R})$), and there is $Y \twoheadrightarrow PX$ with $gf = p_X$ (since $PX$ is Comp-projective). If $Y$ is cover-complete, $f$ is a homeomorphism.

□

Corollary 4.4. Suppose $\text{ED} \subset \mathcal{R} = S\mathcal{R}$. If $X \in \mathcal{R} \cap \text{Comp}_{\text{po}}$, then $(PX, p_X)$ is the unique maximal $\mathcal{R}$-cover of $X$.

Proof. 4.2 (a) and 4.3 (b). □
The following is a qualified converse to 4.2 (a).

**Corollary 4.5.** Suppose that \( ED \subseteq R = S R \). For \( Y \in R \), the following are equivalent.

(a) \( Y \) is ED.
(b) \( Y \) is cover-complete and \( Y \in \text{Comp}_c \).
(c) \( Y \) is cover-complete and \( Y \) has \( e(R) \).

**Proof.**

(a) \( \Rightarrow \) (b): 4.2 (b) and ED \( \subseteq \text{Comp}_c \).
(b) \( \Rightarrow \) (c): By 4.2 (a).
(c) \( \Rightarrow \) (a): By 4.3 (b) (using ED \( \subseteq R \) now), \( (\Pi_Y, p_Y) \) is the unique maximal \( R \)-cover of \( Y \), so if \( Y \) is cover-complete, \( p_Y \) is a homeomorphism. \( \square \)

Here is one (more) triviality valid in (almost) any \( R \).

**Proposition 4.6.** Suppose \( \{0\} \in R \). For any \( X \in R \), with \( |X| \geq 1 \), there are maps \( X \overset{e}{\rightarrow} \{0\} \) (in \( R \)). Such an \( e \) is \( R \)-epi iff \( |X| = 1 \).

**Proof.**

Given such \( e \), there is (the retraction) \( X \overset{r}{\rightarrow} \{0\} \) with \( re = id_{\{0\}} \), so \( e \) is a section. If \( |X| = 1 \), then \( e \) is onto, thus \( R \)-epi. If \( e \) is \( R \)-epi, it becomes an \( R \)-isomorphism, thus a homeomorphism, so \( |X| = 1 \). \( \square \)

5. Epireflectives with Epis not onto, and some maximal covers

First, in summary so far of the situation for \( R \) epireflective in \( \text{Comp} \): If in \( R \), there are epis not onto, then there are no non-void projectives (3.5). That is the case for \( R = \{\emptyset, \{0\}\} \), but here we have the projective (thus unique maximal) covers \( \emptyset \overset{e}{\rightarrow} \emptyset \) and \( \{0\} \overset{e}{\rightarrow} \emptyset \) (3.6). If \( R \) contains the two-point space \( \{0, 1\} \), then \( R \supseteq \text{Comp}_c \) and at least has unique maximal covers for \( X \in \text{Comp}_c \), namely the \( (PX, p_X) \) (4.4).

We now display a large class of such \( R \) with some very strange epis, and non-unique maximal covers. This will be the \( R(E) = SP\{E\} \), for \( E \) as follows.

A space \( E \) in \( \text{Comp} \) will be called strongly rigid if \( |E| \geq 2 \) and the only continuous \( E \overset{e}{\rightarrow} E \) are \( id_E \) and constants. Cook [4] has several of these, including a metric one \( M_1 \).

Note that if \( E \) is strongly rigid, then \( \{0, 1\} \subseteq E \) (since \( |E| \geq 2 \), \( E \) is connected (since a clopen \( U \neq \emptyset \), \( E \) would yield \( E \overset{e}{\rightarrow} \{0, 1\} \overset{e}{\rightarrow} E \), \( |E| \geq c \) (since there are non-constant \( E \overset{e}{\rightarrow} [0, 1] \), using the Tietze-Urysohn Theorem), and \( [0, 1] \not\subseteq E \) (since \( [0, 1] \subseteq E \) would yield non-constant \( E \overset{e}{\rightarrow} [0, 1] \overset{e}{\rightarrow} E \), and \( [0, 1] \) is not strongly rigid).

From Cook’s examples, Trnková [15] and Isbell [12] have shown first, that if \( n \) is any cardinal, there is strongly rigid \( E \) with \( |E| \geq n \), and second, that if there is no measurable cardinal, there is a proper class \( E \) of strongly rigid spaces for which, whenever \( E_1 \neq E_2 \) in \( E \), the only continuous \( E_1 \overset{e}{\rightarrow} E_2 \) are constants (and thus, for \( E_1 \neq E_2 \) in \( E \), neither of \( R(E_1) \) and \( R(E_2) \) contains the other).
Now let $E$ be any strongly rigid space. In the following, terms epi, cover, \ldots refer to $\mathcal{R}(E)$.

Of course 4.4 and 4.5 apply here. On the other hand,

**Proposition 5.1.** Let $F$ be a closed subspace of $E$. Label the inclusion $E \overset{i_F}{\longrightarrow} F$.

(a) $i_F$ is epi iff $|F| > 1$.

(b) If $|F| = 2$, then $(F, i_F)$ is a cover of $E$. If $F \in \text{Comp}_E$ and $(F, i_F)$ is a cover of $E$, then $|F| = 2$.

(In the second part of (b), the supposition ”$F \in \text{Comp}_E$” cannot be dropped, because $E \overset{\text{id}}{\longrightarrow} E$ is a cover.)

**Corollary 5.2.** Any nonconstant $E \overset{g}{\longrightarrow} \{0, 1\}$ is a maximal cover of $E$. Any maximal cover of $E$ is equivalent to one of these. Two of these, with $g$ and $g'$, are equivalent covers of $E$ iff $g(\{0, 1\}) = g'(\{0, 1\})$.

In particular, $(PE, p_E)$ is not a cover of $E$, and there are at least $|E| \geq c$ non-equivalent maximal covers of $E$;

**Proof.** (of 5.1)

(a) By 4.6, if $|F| = 1$, then $i_F$ is not epi. Now suppose $|F| > 1$. Suppose $f, g \in \mathcal{R}(E)$ have common codomain - which might as well be supposed of the form $E^I$ - and $fi_F = gi_F$, i.e., $f|_{i_F} = g|_{i_F}$. Then, for any projection $E^I \overset{\pi_i}{\longrightarrow} E$, we have $\pi_i f|_{i_F} = \pi_i g|_{i_F}$. We want $f = g$, which is equivalent to $\pi_i f = \pi_i g \forall i \in I$.

Let $i \in I$. Then each of $\pi_i f$ and $\pi_i g$ is id$_E$ or constant. If $\pi_i f = \text{id}_E$, then $\pi_i f|_{i_F}$ is not constant (since $|F| \geq 2$), so $\pi_i g|_{i_F}$ is not constant, so $\pi_i g = \text{id}_E$ also. If $\pi_i f$ is constant, say $c$, then $\pi_i f|_{i_F} = c$ also. So $\pi_i g|_{i_F} = c$, and since $|F| \geq 2$, $\pi_i g = c$.

(b) Suppose $|F| = 2$. By (a), $i_F$ is epi. Suppose $i_F f$ is epi. Then $f$ is onto $F$ (since if not, $|\text{range}(f)| = 1$, since $|F| = 2$, but then $|\text{range}(if F)| = 1$ and $i_F f$ is not epi, by 4.6. So $f$ is epi.

Suppose $F \subseteq \text{Comp}_E$. If there are different $p_0, p_1, p_2 \in F$, let $\overline{F} \overset{f}{\longrightarrow} \{0, 1\}$ be $f(i) = p_i$. Then $f$ is not epi (by 4.2 (a)), but $i_F f$ is epi by (a) above.

\[ \square \]

**Proof.** (of 5.2)

If $E \overset{g}{\longrightarrow} \{0, 1\}$ is nonconstant, it is a cover because $F \equiv \{g(0), g(1)\} \overset{g}{\longrightarrow} \{0, 1\}$ is a homeomorphism, and thus a maximal cover because $F$ is cover-complete (being $ED$ 4.2).

Suppose $E \overset{h}{\longrightarrow} Y$ is a maximal cover. Then $h$ is epi, thus nonconstant (6.1). So there are $p_0, p_1 \in Y$ with $h(p_0) \neq h(p_1)$. Define $Y \overset{f}{\longrightarrow} \{0, 1\}$ as $f(i) = p_i$. So $hf$ is a covering-map (by the preceding paragraph), thus $f$ is a
covering map (2.2 (d)). Since \( Y \) is cover-complete, \( f \) is a homeomorphism, so \((Y, h) \) and \((\{0, 1\}, hf)\) are equivalent.

Now suppose \( E_\phi \) are non-constant. There are two homeomorphisms \( h \) of \( \{0, 1\} \), the identity and ”interchange 0 and 1”. And, range \((g) = range(g') \) iff \( g' = gh \) for one of these \( h \).

\[ \square \]

Remark 5.3. Cook’s specific strongly rigid \( M_1 \) has these further features:

\( M_1 \) has a countable infinity of disjoint subcontinua; if \( K \) is any proper subcontinuum of \( M_1 \), the only maps \( M_1 \leftarrow \leftarrow \) in \( \mathbb{R} \) are inclusion and constants. (See [4]). Then in the category \( \mathcal{R}(M_1) \), in 5.1 and 5.2, \( E = M_1 \) may be replaced by any proper subcontinuum \( K \) of \( M_1 \) (as the proofs there show).

6. AN APPLICATION TO LATTICE-ORDERED GROUPS

We now convert the situations of maximal covers in \( \mathcal{R} \subset \text{Comp} \) to situations of maximal essential extensions in subcategories of a category of algebras. We use terminology categorically dual to the items in 2.1 (a) - (e), respectively, namely (a) monic, (b) essential extension, (c) essentially complete, maximal essential extension (or, essential completion), (d) injective, injective hull, (e) injectively complete.

The category of algebras is \( W^* \), the category of archimedean lattice-ordered groups with distinguished strong order unit, and \( \ell \)-group homomorphisms carrying unit to unit. \( W^* \) has monics one-to-one, and is injectively complete; see [3]. Consequently, the dual of 2.7 applies to \( W^* \).

For \( X \in \text{Comp} \), the continuous real-valued functions \( C(X) \), with unit the constant function 1, is a \( W^* \)-object, and we have the functor \( W^* \text{Comp} \): for \( X \rightarrow Z \) in \( \text{Comp} \), \( C(X) \xrightarrow{C(f)} C(Z) \) is \( C(f) = f \circ \tau \). This has a left adjoint, the Yosida functor: For each \( G \in W^* \), there is \( YG \in \text{Comp} \) and \( G \xrightarrow{\varphi} C(YG) \) monic in \( W^* \); for each \( G \xrightarrow{\varphi} H \) in \( W^* \), there is unique \( YG \xrightarrow{Y \varphi} YH \) in \( \text{Comp} \) ”realizing \( \varphi \)” as \( \varphi(g) = g \circ Y \varphi \). Note that \( YC(X) \simeq X \), and that \( \varphi \) is one-to-one iff \( Y \varphi \) is onto. (See [10]).

Basic features of \((Y, C)\), and some diagram-chasing, convert the situations in \( \text{Comp} \) discussed in previous sections to ”dual” situations in \( W^* \), as follows. (We omit the calculations).

Suppose \( \mathcal{R} \) is epireflective in \( \text{Comp} \), and \( \{0, 1\} \in \mathcal{R} \) (so \( \text{Comp}_0 \in \mathcal{R} \)). For brevity, set \( *\mathcal{R} = \{ G \in W^*|YG \in \mathcal{R} \} \).

**Proposition 6.1.**

(a) \( *\mathcal{R} \) is monomcoreflective in \( W^* \).

(b) \( C(X) \xrightarrow{\varphi} H \) is monic in \( *\mathcal{R} \) iff \( X \xrightarrow{Y \varphi} YH \) is epi in \( \mathcal{R} \).

(c) \( *\mathcal{R} \) has an injective other than \( \{0\} \) iff monics in \( *\mathcal{R} \) are one-to-one iff \( \mathcal{R} \)-epis are onto. When this occurs, \( *\mathcal{R} \) is injective-complete, with injective hulls \( G \xrightarrow{C(YG)} C(P(YG)) \).

(d) If \( X \) is ED, then \( C(X) \) is essentially complete in \( *\mathcal{R} \).
(e) If \( X \in \text{Comp}_C \), then \( C(D) \xrightarrow{C_p X} C(PX) \) is the unique maximal essential extension of \( C(X) \) in \( ^*R \).

Now consider, as in §5, strongly rigid \( E \in \text{Comp} \) and its generated epireflective \( R(E) \). By 5.1 and 6.1(b), \( ^*R(E) \) has monics which are not one-to-one, and thus no \( \not\in \{0\} \) injectives. 6.1(d) and (e) hold in \( ^*R(E) \).

Note that \( \{0,1\} \in \text{Comp} \) has \( C(\{0,1\}) = \mathbb{R}^2 \in W^* \), the self-homeomorphisms of \( \{0,1\} \) are the identity and "interchange points", and these correspond to the only self-isomorphisms of \( \mathbb{R}^2 \), which are the identity, and \( H(x,y) = (y,x) \).

From 5.2 we obtain

**Corollary 6.2.** In \( ^*R(E) \), the maximal essential extensions of \( C(E) \) are exactly the \( W^* \)-surjections \( C(E) \xrightarrow{\varphi} \mathbb{R}^2 \). Two of these, \( \varphi \) and \( \varphi' \), are equivalent iff either \( \varphi = \varphi' \), or \( \varphi' = \varphi H \).

**REFERENCES**


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