The combinatorial derivation

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Abstract

Let $G$ be a group, $P_G$ be the family of all subsets of $G$. For a subset $A \subseteq G$, we put $\Delta(A) = \{ g \in G : |gA \cap A| = \infty \}$. The mapping $\Delta : P_G \rightarrow P_G, A \mapsto \Delta(A)$, is called a combinatorial derivation and can be considered as an analogue of the topological derivation $d : P_X \rightarrow P_X, A \mapsto A^d$, where $X$ is a topological space and $A^d$ is the set of all limit points of $A$. Content: elementary properties, thin and almost thin subsets, partitions, inverse construction and $\Delta$-trajectories, $\Delta$ and $d$.


Keywords: Combinatorial derivation; $\Delta$-trajectories; large, small and thin subsets of groups; partitions of groups; Stone-Čech compactification of a group.

1. Introduction

Let $G$ be a group with the identity $e$, $P_G$ be the family of all subsets of $G$. For a subset $A$ of $G$, we denote

$$\Delta(A) = \{ g \in G : |gA \cap A| = \infty \},$$

observe that $\Delta(A) \subseteq AA^{-1}$, and say that the mapping

$\Delta : P_G \rightarrow P_G, A \mapsto \Delta(A)$

is the combinatorial derivation.

In this paper, on one hand, we analyze from the $\Delta$-point of view a series of results from Subset Combinatorics of Groups (see the survey [9]), and point out some directions for further progress. On the other hand, the $\Delta$-operation is interesting and intriguing by its own sake. In contrast to the trajectory $A \mapsto$
The ∆-trajectory $A \rightarrow \Delta(A) \rightarrow \Delta^2(A) \rightarrow \ldots$ of a subset $A$ of $G$ could be surprisingly complicated: stabilizing, increasing, decreasing, periodic or chaotic. For a symmetric subset $A$ of $G$ with $e \in A$, there exists a subset $X \subseteq G$ such that $\Delta(X) = A$. We conclude the paper by demonstrating how $\Delta$ and a topological derivation $d$ arise from some unified ultrafilter construction.

We note also that $\Delta(A)$ may be considered as some infinite version of the symmetry sets well-known in Additive Combinatorics \[11\] , p. 84]. Given a finite subset $A$ of an Abelian group $G$ and $\alpha \geq 0$, the symmetry set $\text{Sym}_\alpha(A)$ is defined by

$$\text{Sym}_\alpha(A) = \{g \in G : |A \cap (A + g)| \geq \alpha |A|\}.$$

2. Elementary properties

Claim 2.1. $(\Delta(A))^{-1} = \Delta(A)$, $\Delta(A) \subseteq AA^{-1}$.

Claim 2.2. $\Delta(A) = \emptyset \iff e \notin \Delta(A) \iff A$ is finite.

Claim 2.3. For subsets $A, B$ of $G$, we let

$$\Delta(A, B) = \{g \in G : |gA \cap B| = \infty\}$$

and note that

$$\Delta(A \cup B) = \Delta(A) \cup \Delta(B) \cup \Delta(A, B),$$

$$\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$$

Claim 2.4. If $F$ is a finite subset of $G$ then

$$\Delta(FA) = F\Delta(A)F^{-1}.$$

Claim 2.5. If $A$ is an infinite subgroup then $A = \Delta(A)$ but the converse statement does not hold, see Theorem 6.2.

3. Thin and almost thin subsets

A subset $A$ of a group $G$ is said to be \[8\]:

- thin if either $A$ is finite or $\Delta(A) = \{e\}$;
- almost thin if $\Delta(A)$ is finite;
- $k$-thin ($k \in \mathbb{N}$) if $|gA \cap A| \leq k$ for each $g \in G \setminus \{e\}$;
- sparse if, for every infinite subset $X \subseteq G$, there exists a non-empty finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is infinite;
- $k$-sparse ($k \in \mathbb{N}$) if, for every infinite subset $X \subseteq G$, there exists a subset $F \subset X$ such that $|F| \leq k$ and $\bigcap_{g \in F} gA$ is finite.

The following statements are from \[8\].

Theorem 3.1. Every almost thin subset $A$ of a group $G$ can be partitioned in $3^{\lfloor \Delta(A) \rfloor - 1}$ thin subsets. If $G$ has no elements of odd order, then $A$ can be partitioned in $2^{\lfloor \Delta(A) \rfloor - 1}$ thin subsets.
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**Theorem 3.2.** A subset $A$ of a group $G$ is 2-sparse if and only if $X^{-1}X \notin \Delta(A)$ for every infinite subset $X$ of $G$.

**Theorem 3.3.** For every countable thin subset $A$ of a group $G$, there is a thin subset $B$ such that $A \cup B$ is 2-sparse but not almost thin.

**Theorem 3.4.** Suppose that a group $G$ is either torsion-free or, for every $n \in \mathbb{N}$, there exists a finite subgroup $H_n$ of $G$ such that $|H_n| > n$. Then there exists a 2-sparse subset of $G$ which cannot be partitioned in finitely many thin subsets.

By Theorem 3.2, every almost thin subset is 2-sparse. By Theorems 3.3, 3.4, the class of 2-sparse subsets is wider than the class of almost thin subsets. By Theorem 3.3, a union of two thin subsets needs not to be almost thin. By Theorem 2.3, a union $A_1 \cup \ldots \cup A_n$ of almost thin subset is almost thin if and only if $\Delta(A_i, A_j)$ is finite for all $i, j \in \{1, \ldots, n\}$, By Claim 2.4, if $A$ is almost thin and $K$ is finite then $KA$ is almost thin.

The following statements are from [7].

**Theorem 3.5.** For every infinite group $G$, there exists a 2-thin subset such that $G = XX^{-1} \cup X^{-1}X$.

**Theorem 3.6.** For every infinite group $G$, there exists a 4-thin subset such that $G = XX^{-1}$.

Since $\Delta(X) = \{e\}$ for each infinite thin subset of $G$, Theorem 3.6 gives us $X$ with $\Delta(X) = \{e\}$ and $XX^{-1} = G$.

4. Large and small subsets

A subset $A$ of a group $G$ is called [8]:

- **large** if there exists a finite subset $F$ of $G$ such that $G = FA$;
- **$\Delta$-large** if $\Delta(A)$ is large;
- **small** if $(G \setminus A) \cap L$ is large for each large subset $L$ of $G$;
- **$P$-small** if there exists an injective sequence $(g_n)_{n \in \omega}$ in $G$ such that the subsets $\{g_nA : n \in \omega\}$ are pairwise disjoint;
- **almost $P$-small** if there exists an injective sequence $(g_n)_{n \in \omega}$ in $G$ such that the family $\{g_nA : n \in \omega\}$ is almost disjoint, i.e. $g_nA \cap g_mA$ is finite for all distinct $n, m \in \omega$;
- **weakly $P$-small** if, for every $n \in \omega$, one can find distinct elements $g_1, \ldots, g_n$ of $G$ such that the subsets $g_1A, \ldots, g_nA$ are pairwise disjoint.

Let $G$ be a group, $A$ is a large subset of $G$. We take a finite subset $F$ of $G$, $F = \{g_1, \ldots, g_n\}$ such that $G = FA$. Take an arbitrary $g \in G$. Then $g_iA \cap gA$ is infinite for some $i \in \{1, \ldots, n\}$, so $g_i^{-1}g \in \Delta(A)$. Hence, $G = F\Delta(A)$ and $A$ is $\Delta$-large. By Theorem 3.6, the converse statement is very far from being true.

If $A$ is not small then $FA$ is thick (see Definition 5.2) for some finite subset $F$. It follows that $\Delta(FA) = G$. By Claim 2.4, $\Delta(FA) = F\Delta(A)F^{-1}$, so if $G$ is Abelian then $A$ is $\Delta$-large.
J. Erde proved that every non-small subset of an arbitrary infinite non-Abelian group $G$ is $\Delta$-large.

It is easy to see that $A$ is P-small (almost P-small) if and only if there exists an infinite subset $X$ of $G$ such that $X^{-1}X \cap PP^{-1} = \{e\}$ ($X^{-1}X \cap \Delta(X) = \{e\}$). $A$ is weakly P-small if and only if, for every $n \in \omega$, there exists $F \subseteq G$ such that $|F| = n$ and $F^{-1}F \cap PP^{-1} = \{e\}$.

By [8, Lemma 4.2], if $A$ is not large then $A$ is small and P-small. Using the inverse construction from Section 6, we can find $A$ such that $A$ is not $\Delta$-large and $A$ is not P-small.

Every infinite group $G$ has a weakly P-small not P-small subsets [1]. Moreover, $G$ has almost P-small not P-small subset and, if $G$ is countable, weakly P-small not almost P-small subset. By [8], every almost P-small subset can be partitioned in two P-small subsets. If $A$ is either almost or weakly P-small then $G \setminus \Delta(A)$ is infinite, but a subset $A$ with infinite $G \setminus \Delta(A)$ could be large: $G = \mathbb{Z}$, $A = 2\mathbb{Z}$.

5. Partitions

Let $G$ be a group and let $G = A_1 \cup \ldots \cup A_n$ be a finite partition of $G$. In section 7, we show that at least one cell $A_i$ is $\Delta$-large, in particular, $A_iA_i^{-1}$ is large. If $G$ is infinite amenable group and $\mu$ is a left invariant Banach measure on $G$, we can strengthen this statement: there exist a cell $A_i$ and a finite subset $F$ such that $|F| \leq n$ and $G = F\Delta(A_i)$. To verify this statement, we take $A_i$ such that $\mu(A_i) \geq \frac{1}{n}$ and choose distinct $g_1, \ldots , g_m$ such that $\mu(g_kA_i \cap g_lA_i) = 0$ for all distinct $k, l \in \{1, \ldots , m\}$, and the family $\{g_1A_i, \ldots , g_mA_i\}$ is maximal with respect to this property. Clearly, $m \leq n$. For each $g \in G$, we have $\mu(gA_i \cap g_kA_i) > 0$ for some $k \in \{1, \ldots , m\}$ so $g_k^{-1}g \in \Delta(A_i)$ and $G = \{g_1, \ldots , g_m\}\Delta(A_i)$.

By [10, Theorem 12.7], for every partition $A_1 \cup \ldots \cup A_n$ of an arbitrary group $G$, there exist a cell $A_i$ and a finite subset $F$ of $G$ such that $G = FA_iA_i^{-1}$ and $|F| \leq 2^{2^{n-1}-1}$.

S. Slobodianuk strengthened this statement: there are $F$ and $A_i$ such that $|F| \leq 2^{2^{n-1}-1}$ and $G = F\Delta(A_i)$.

It is an old unsolved problem [5, Problem 13.44] whether $i$ and $F$ can be chosen so that $G = FA_iA_i^{-1}$ and $|F| \leq n$, see also [10, Question 12.1].

**Question 5.1.** Given any partition $G = A_1 \cup \ldots \cup A_n$, do there exist $F$ and $A_i$ such that $G = F\Delta(A_i)$ and $|F| \leq 2^n$?

**Definition 5.2.** A subset $A$ of a group $G$ is called [11]:

- thick if $G \setminus A$ is not large;
- $k$-prethick ($k \in \mathbb{N}$) if there exists a subset $F$ of $G$ such that $|F| \leq k$ and $FA$ is thick;
- prethick if $A$ is $k$-prethick for some $k \in \mathbb{N}$.

By [3, Theorem 5.3.2], for a group $G$, the following two conditions (i) and (ii) are equivalent:
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(i) for every partition $G = A \cup B$, either $G = AA^{-1}$ or $G = BB^{-1}$;
(ii) each element of $G$ has odd order.
If $G$ is infinite, we can show that these conditions are equivalent to
(iii) for every partition $G = A \cup B$, either $G = \Delta(A)$ or $G = \Delta(G)$.

6. INVERSE CONSTRUCTION AND $\Delta$-TRAJECTORIES

**Theorem 6.1.** Let $G$ be an infinite group, $A \subseteq G$, $A = A^{-1}$, $e \in A$. Then there exists a subset $X$ of $G$ such that $\Delta(X) = A$.

**Proof.** First, assume that $G$ is countable and write the elements of $A$ in a list $\{a_n : n < \omega\}$, if $A$ is finite then all but finitely many $a_n$ are equal to $e$. We represent $G \setminus A$ as a union $G \setminus A = \bigcup_{n \in \omega} B_n$ of the finite subsets such that $B_n \subseteq B_{n+1}$, $B_{n+1}^{-1} = B_n$. Then we choose inductively a sequence $(X_n)_{n \in \omega}$ of finite subsets of $G$

$$X_n = \{x_{n0}, x_{n1}, \ldots, x_{nn}, a_0x_{n0}, \ldots, a_nx_{nn}\}$$

such that $X_nX_n^{-1} \cap B_n = \{e\}$ for all $m \leq n < \infty$.

After $\omega$ steps, we put $X = \bigcup_{n \in \omega} X_n$. By the construction, $\Delta(X) = A$.

If $|A| \leq \aleph_0$ but $G$ is not countable, we take a countable subgroup $H$ of $G$ such that $A \subseteq H$, forget about $G$ and find a subset $X \subseteq H$ such that $\Delta(X)$ is equal to $A$ in $H$. Since $gA \cap A = \varnothing$ for each $g \in G \setminus H$, we have $\Delta(X) = A$.

At last, let $|A| > \aleph_0$. By above paragraph, we may suppose that $|A| = |G|$. We enumerate $A = \{a_\alpha : \alpha < |G|\}$ and construct inductively a sequence $(X_\alpha)_{\alpha < |G|}$ of finite subsets of $G$ and an increasing sequence $(H_\alpha)_{\alpha < |G|}$ of subgroup of $G$ such that if $\alpha = 0$ or $\alpha$ is a limit ordinal, $n \in \omega$,

$$X_{\alpha+n} = \{x_{\alpha+n,0}x_{\alpha+n,1}, \ldots, x_{\alpha+n,m}, a_0x_{\alpha+n,0}, \ldots, a_{\alpha+n}x_{\alpha+n,m}\},$$

$$X_{\alpha+n} \subseteq H_{\alpha+n+1} \setminus H_{\alpha+n}, X_{\alpha+n}X_{\alpha+n}^{-1} \subseteq A \cup (H_{\alpha+n+1} \setminus H_{\alpha+n}).$$

After $|G|$ steps, we put $X = \bigcup_{\alpha < |G|} X_\alpha$. By the construction, $\Delta(X) = A$. \qed

Let $A_1, \ldots, A_m$ be subsets of an infinite group $G$ such that $G = A_1 \cup \ldots \cup A_m$. By the Hindman theorem [4, Theorem 5.8], there are exists $i \in \{1, \ldots, m\}$ and an injective sequence $(g_n)_{n \in \omega}$ in $G$ such that $FP(g_n)_{n \in \omega} \subseteq A_i$, where $FP(g_n)_{n \in \omega}$ is a set of all element of the form $g_{i_1}g_{i_2} \ldots g_{i_k}$, $i_1 < \ldots, i_k < \omega$, $k \in \omega$.

We show that there exists $X \subseteq FP(g_n)_{n \in \omega}$ such that $\Delta(X) = \{e\} \cup FP(g_n)_{n \in \omega} \cup (FP(g_n)_{n \in \omega})^{-1}$. We note that if $G$ is countable, at each step $n$ of the inverse construction, the elements $x_{n0}, \ldots, x_{nn}$ can be chosen from any previous infinite subset $Y$ of $G$. We enumerate $FP(g_n)_{n \in \omega}$ in a sequence $(a_n)_{n \in \omega}$ and put $Y = \{g_n : n \in \omega\}$. Using above observation, we get the desired $X$.

If $G$ is countable, we can modify the inverse construction to get $X$ such that $\Delta(X) = A$ and $|X \cap g_1g_2X| < \infty$ for all distinct $g_1, g_2 \in G \setminus \{e\}$, in particular, $X$ is 3-sparse and, in particular, small.
Another modification, we can choose $X$ such that $X \cap gX \neq \emptyset$ for each $g \in G$. If we take $A$ not large, then we get $X$ which is not P-small and $X$ is not $\Delta$-large, see Section 4.

**Theorem 6.2.** Let $G$ be a countable group such that, for each $g \in G \setminus \{ e \}$, the set $\sqrt{g} = \{ x \in G : x^2 = g \}$ is finite. Then the following statements hold:

1. Given any subset $X_0 \subseteq G$, $X_0 = X_0^{-1}$, $e \in X_0$, there exists a sequence $(X_n)_{n \in \omega}$ of subsets of $G$ such that $\Delta(X_{n+1}) = X_n$ and $X_m \cap X_n = \{ e \}$, $0 < m < n < \omega$.

2. There exists a sequence $(X_n)_{n \in \omega}$ of subsets of $G$ such that $\Delta(X_n) = X_{n+1}$, $X_m \cap X_n = \{ e \}$, $m, n \in \mathbb{Z}, m \neq n$.

3. There exists a subset $A$ of $G$ such that $\Delta(A) = A$ but $A$ is not a subgroup.

4. There exists a subset $A$ such that $A \supset \Delta(A) \supset \Delta^2(A) \supset \ldots$.

5. There exists a subset $A$ such that $A \supset \Delta(A) \supset \Delta^2(A) \supset \ldots$.

6. For each natural number $n$, there exists a periodic $\Delta$-trajectory $X_0, \ldots, X_{n-1}$ of length $n$: $X_1 = \Delta(X_0), X_2 = \Delta(X_1), \ldots, X_n = \Delta(X_{n-1})$ such that $X_i \cap X_j = \{ e \}$, $i < j < n$.

**Proof.** We use the following simple observation

(*) if $F$ is a finite subset of an infinite group $G$ and $g \notin F$ then the set \( \{ x \in G : x^{-1}gx \notin F \} \) is infinite.

In constructions of corresponding trajectories, at each inductive step, we use a finiteness of $\sqrt{g}$ and (*) in the following form:

(**) if $a \in G$, $F$ is a finite subset of $G$, $F \cap \{ e, a^{\pm 1} \} = \emptyset$ then there exists $x \in G$ such that

\[
\{ x^{\pm 1}, (ax)^{\pm 1} \} \cap F = \emptyset.
\]

We show how to get a 2-periodic trajectory: $X, Y, \Delta(X) = Y, \Delta(Y) = X, X \cap Y = \{ e \}$. We write $G$ as a union $G = \bigcup_{n \in \omega} F_n$ of increasing chain \( \{ F_n : n \in \omega \} \) of finite symmetric subsets $F_0 = \{ e \}$. We put $X_0 = Y_0 = \{ e \}$ and construct inductively with usage of (**), two chains $(X_n)_{n \in \omega}$, $(Y_n)_{n \in \omega}$ of finite subsets of $G$ such that, for each $n \in \omega$,

\[
X_{n+1} = \{ (x(y))^{\pm 1}, (yx(y))^{\pm 1} : y \in Y_0 \cup \ldots \cup Y_n \},
\]

\[
Y_{n+1} = \{ (g(x))^{\pm 1}, (xg(x))^{\pm 1} : x \in X_0 \cup \ldots \cup X_n \},
\]

\[
(X_0 \cup \ldots \cup X_n) \cap (Y_0 \cup \ldots \cup Y_n) = \{ e \},
\]

\[
X_{n+1} \cap (F_{n+1} \setminus (Y_0 \cup \ldots \cup Y_n)) = \emptyset,
\]

\[
(Y_0 \cup \ldots \cup Y_n) \cap (F_{n+1} \setminus (X_0 \cup \ldots \cup X_n)) = \emptyset,
\]

\[
(Y_0 \cup \ldots \cup Y_n) \cap (F_{n+1} \setminus (X_0 \cup \ldots \cup X_n)) = \emptyset.
\]

After $\omega$ steps, we put $X = \bigcup_{n \in \omega} X_n, Y = \bigcup_{n \in \omega} Y_n$. □
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7. $\Delta$ and $d$

For a subset $A$ of a topological space $X$, the subset $A^d$ of all limit points of $A$ is called a derived subset, and the mapping $d : \mathcal{P}(X) \to \mathcal{P}(X)$, $A \to A^d$, defined on the family of $\mathcal{P}(X)$ of all subsets of $X$, is called the topological derivation, see [6, §9].

Let $X$ be a discrete set, $\beta X$ be the Stone-Čech compactification of $X$. We identify $\beta X$ with the set of all ultrafilters on $X$, $X$ with the set of all principal ultrafilters, and denote $X^* = \beta X \setminus X$ the set of all free ultrafilters. The topology of $\beta X$ can be defined by the family $\{A : A \subseteq X\}$ as a base for open sets, $\overline{A} = \{p \in \beta X : A \in p\}$, $A^* = A \cap G^*$. For a filter $\varphi$ on $X$, we put $\overline{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$, $\varphi^* = \overline{\varphi} \cap G^*$.

Let $G$ be a discrete group, $p \in \beta G$. Following [2, Chapter 3], we denote

$$cl(A, p) = \{g \in G : A \in gp\}, \quad gp = \{gP : P \in p\},$$

say that $cl(A, p)$ is a closure of $A$ in the direction of $p$, and note that

$$\Delta(A) = \bigcap_{p \in A^*} cl(A, p).$$

A topology $\tau$ on a group $G$ is called left invariant if the mapping $l_g : G \to G$, $l_g(x) = gx$ is continuous for each $g \in G$. A group $G$ endowed with a left invariant topology $\tau$ is called left topological. We note that a left invariant topology $\tau$ on $G$ is uniquely determined by the filter $\varphi$ of neighbourhoods of the identity $e \in G$, $\overline{\varphi}$ and $\varphi^*$ are the sets of all ultrafilters an all free ultrafilters of $G$ converging to $e$. For a subset $A$ of $G$, we have

$$A^d = \bigcap_{p \in (\tau^*)} cl(A, p),$$

and note that $A^d \subseteq \Delta(A)$ if $A$ is a neighbourhood of $e$ in $(G, \tau)$.

Now we endow $G$ with the discrete topology and, following [4, Chapter 4], extend the multiplication on $G$ to $\beta G$. For $p, q \in \beta G$, we take $P \in p$ and, for each $g \in P$, pick some $Q_g \in q$. Then $\bigcup_{g \in P} gQ_g \subseteq pq$ and each member of $pq$ contains a subset of this form. With this multiplication, $\beta G$ is a compact right topological semigroup. The product $pq$ can also be defined by the rule [2, Chapter 3]:

$$A \subseteq G, \quad A \in pq \Leftrightarrow cl(A, q) \in p.$$

If $(G, \tau)$ is left topological semigroup then $\tau$ is a subsemigroup of $\beta G$. If an ultrafilter $p \in \tau$ is taken from the minimal ideal $K(\tau)$ of $\tau$, by [2, Theorem 5.0.25], there exists $P \in p$ and finite subset $F$ of $G$ such that $Fcl(P, p)$ is neighbourhood of $e$ in $\tau$. In particular, if $\tau$ indiscrete ($\tau = \{\varnothing, G\}$), $p \in K(\beta G)$ and $P \in p$ then $cl(P, p)$ is large. If $G$ is infinite, $p \in K(\beta G)$ is free, so $cl(P, p) \subseteq \Delta(P)$ and $P$ is $\Delta$-large. If a group $G$ is finitely partitioned $G = A_1 \cup \ldots \cup A_n$, then some cell $A_i$ is a member of $p$, hence $A_i$ is $\Delta$-large.
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