Closed ideals in the functionally countable subalgebra of $C(X)$

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Abstract

In this paper, closed ideals in $C_c(X)$, the functionally countable subalgebra of $C(X)$, with the $m_c$-topology are studied. We show that if $X$ is a CUC-space, then $C^*_c(X)$ with the uniform norm-topology is a Banach algebra. Closed ideals in $C_c(X)$ as a modified countable analogue of closed ideals in $C(X)$ with the $m$-topology, are characterized. For a zero-dimensional space $X$, we show that a proper ideal in $C_c(X)$ is closed if and only if it is an intersection of maximal ideals of $C_c(X)$. It is also shown that every ideal in $C_c(X)$ with the $m_c$-topology is closed if and only if it is a $P$-space if and only if every ideal in $C(X)$ with the $m$-topology is closed. Also, for a strongly zero-dimensional space $X$, it is proved that every properly closed ideal in $C^*_c(X)$ is an intersection of maximal ideals of $C^*_c(X)$ if and only if $X$ is pseudocompact if and only if every properly closed ideal in $C^*(X)$ is an intersection of maximal ideals of $C^*(X)$. Finally, we show that if $X$ is a $P$-space, then the family of $e_c$-ultrafilters and $z_c$-ultrafilter coincide.

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1. Introduction

In what follows $X$ stands for an infinite completely regular Hausdorff topological space (i.e., infinite Tychonoff space) and $C(X)$ as usual denotes the ring of all real-valued continuous functions on $X$. $C^*(X)$ designates the subring of $C(X)$ containing all those members which are bounded over $X$. For
each \( f \in C(X) \), the zero-set of \( f \), denoted by \( Z(f) \), is the set of zeros of \( f \) and \( X \setminus Z(f) \) is the cozero-set of \( f \) and the set of all zero-sets in \( X \) is denoted by \( Z(X) \). An ideal \( I \) in \( C(X) \) is called a z-ideal if \( f \in I \), \( g \in C(X) \) and \( Z(f) \subseteq Z(g) \), then \( g \in I \). The space \( \beta X \) is the Stone-\( Č \)ech compactification of \( X \) and for any \( p \in \beta X \), the maximal ideal \( M^p \) of \( C(X) \) is the set of all \( f \in C(X) \) for which \( p \in \text{cl}_X Z(f) \). Moreover, \( M^p \) is fixed if and only if \( p \in X \) (in which case, we put \( M^p = M_p = \{ f \in C(X) : p \in Z(f) \} \) ). Whenever \( \frac{C(X)}{M^X} \cong \mathbb{R} \), then \( M^p \) is called real, else hyper-real, see [5, Chapter 8]. We recall that a zero-dimensional space is a Hausdorff space with a base consisting of clopen (closed-open) sets. A Tychonoff space \( X \) is called strongly zero-dimensional if for every finite cover \( \{ U_i \}_{i=1}^k \) of \( X \) by cozero-sets there exists a finite refinement \( \{ V_i \}_{i=1}^m \) of mutually disjoint open sets. A Tychonoff space \( X \) is strongly zero-dimensional if and only if \( \beta X \) is zero-dimensional, see [2].

The subring of \( C(X) \) consisting of those functions with countable (resp. finite) image, which is denoted by \( C_c(X) \) (resp. \( C^f(X) \)) is an \( \mathbb{R} \)-subalgebra of \( C(X) \). The subring \( C^*_c(X) \) of \( C_c(X) \) consists of bounded elements of \( C_c(X) \). So \( C^*_c(X) = C^*(X) \cap C_c(X) \). The rings \( C_c(X) \) and \( C^f(X) \) are introduced and investigated in [3] and more studied in [1], [4], [9], [10] and [12]. A topological space \( X \) is called countably pseudocompact, briefly, \( c \)-pseudocompact if \( C_c(X) = C^*_c(X) \). A nonempty subfamily \( F \) of \( Z_c(X) := \{ Z(f) : f \in C_c(X) \} \) is called a \( z_c \)-filter if it is a filter on \( X \). For an ideal \( I \) in \( C_c(X) \) and a \( z_c \)-filter \( F \), we define \( Z_c[I] = \{ Z(f) : f \in I \} \) and \( Z_c^{-1}[F] = \{ f \in C_c(X) : Z(f) \in F \} \). It is observed that \( F = Z_c[Z_c^{-1}[F]] \). Also, \( Z_c[I] \) is a \( z_c \)-filter on \( X \) and \( Z_c^{-1}[Z_c[I]] \supseteq I \). If the equality holds, then \( I \) is called a \( z_c \)-ideal. This means that if \( f \in I \), \( g \in C_c(X) \) and \( Z(f) \subsetneq Z(g) \), then \( g \in I \). So maximal ideals in \( C_c(X) \) are \( z_c \)-ideals. In the same way, for an ideal \( I \) of \( C^*_c(X) \) and a \( z_c \)-filter \( F \) on \( X \), \( E_c(I) \) is an \( e_c \)-filter and \( E_c^{-1}(F) \) is an \( e_c \)-ideal. The counterpart notions are \( E_c^{-1}(E_c(I)) \supseteq I \) and \( E_c(E_c^{-1}(F)) = F \), see [14]. By \( \beta_0 X \), we mean the Banaschewski compactification of a zero-dimensional space \( X \). If \( \beta X \) is zero-dimensional, then \( \beta X = \beta_0 X \), see [13, Section 4.7] for more details. According to [1, Theorems 4.2, 4.8], for any \( p \in \beta_0 X \), the maximal ideal \( M^p \) of \( C_c(X) \) is the set of all \( f \in C_c(X) \) for which \( p \in \text{cl}_{\beta_0 X} Z(f) \), or equivalently, it is the set of all \( f \in C_c(X) \) for which \( \pi_p \in \text{cl}_{\beta_0 X} Z(f) \). Moreover, \( M^p \) is fixed if and only if \( p \in X \) (in which case, we put \( M^p = M_p = \{ f \in C_c(X) : p \in Z(f) \} \). Let \( S \) be a subring of \( C(X) \) and a topological space. An ideal \( I \) of \( S \) is called a closed ideal if \( I = \text{cl}_S I \), briefly, \( I = \text{cl} I \). The paper is organized as follows. In Section 2, we introduce the \( m_c \)-topology on \( C_c(X) \) and derive some corollaries on the ideals of \( C_c(X) \) and \( C^*_c(X) \). We show that if \( X \) is a CUC-space, then \( C^*_c(X) \) with the uniform-norm topology is a Banach algebra. It is shown that an ideal in \( C_c(X) \) is a z-ideal if and only if it is a \( z_c \)-ideal. In [5], closed ideals in \( C(X) \) with the \( m \)-topology are characterized. In Section 3, the countable analogue of this characterization is given. We show that a proper ideal in \( C_c(X) \) is closed if and only if it is an intersection of maximal ideals in \( C_c(X) \). It is also shown that every ideal...
in $C_c(X)$ is closed if and only if $X$ is a $P$-space if and only if every ideal in $C(X)$ is closed. For a strongly zero-dimensional space $X$, we prove that every properly closed ideal in $C^*_c(X)$ is an intersection of maximal ideals of $C^*_c(X)$ if and only if $X$ is pseudocompact if and only if every proper closed ideal in $C^*_c(X)$ is an intersection of maximal ideals of $C^*_c(X)$. Finally, we show that if $X$ is a $P$-space, then the family of $e_c$-ultrafilters and $z_c$-ultrafilter coincide.

2. SOME PROPERTIES OF IDEALS IN $C_c(X)$

The $m$-topology on $C(X)$ was first introduced and studied by Hewitt [8], the generalizing work of E. H. Moore. In his article, he demonstrated that certain classes of topological spaces $X$ can be characterized by topological properties of $C(X)$ with the $m$-topology. For example, he showed that $X$ is pseudocompact if and only if $C(X)$ with the $m$-topology is first countable. Several authors have investigated the topological properties of $X$ via properties of $C(X)$, for more information, one can refer to [6] and [11]. The $m$-topology on $C(X)$ is defined by taking the sets of the form

$$B(f, u) = \{ g \in C(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X \},$$

as a base for the neighborhood system at $f$, for each $f \in C(X)$ and each positive unit $u$ of $C(X)$. The $m_c$-topology (in brief, $m_c$) on $C_c(X)$ is determined by considering the sets of the form

$$B(f, u) = \{ g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X \},$$

as a base for the neighborhood system at $f$, for each $f \in C_c(X)$ and each positive unit $u$ of $C_c(X)$. The uniform topology, or the $u_c$-topology (in brief, $u_c$) on $C_c(X)$ is defined by taking the sets of the form

$$B(f, \varepsilon) = \{ g \in C_c(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in X \},$$

as a base for the neighborhood system at $f$, for each $f \in C_c(X)$ and each $\varepsilon > 0$. Equivalently, a base at $f$ is given by all sets

$$B(f, u) = \{ g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X \},$$

where $u$ is a positive unit of $C^*_c(X)$. We observe that $u_c \subseteq m_c$. It is shown in [15] that $u_c = m_c$ if and only if $X$ is countably pseudocompact. The $u_c$-topology turns $C^*_c(X)$ into a metric space with $d(f, g) = ||f - g|| = \sup \{|f(x) - g(x)| : x \in X \}$. Also, the $m_c$-topology is contained in the relative $m$-topology.

We remind a well-known result that due to Rudin, Pełczyński and Semadeni which asserts that a compact Hausdorff space $X$ is functionally countable (i.e., $C(X) = C_c(X)$) if and only if $X$ is scattered. So if $X$ is a compact scattered space or a countable space, then $C(X) = C_c(X)$, and thus the $m_c$-topology and the $m$-topology coincide.

**Proposition 2.1.** Let $I$ be an ideal in $C_c(X)$ (resp. $C^*_c(X)$) and the topology on $C_c(X)$ be the $m_c$-topology. Then:

(i) $cI$ is an ideal in $C_c(X)$ (resp. $C^*_c(X)$) and hence $I$ is contained in a closed ideal.

(ii) If $I$ is a proper ideal, then $cI$ is also a proper ideal and hence there is no proper dense ideal in $C_c(X)$ (resp. $C^*_c(X)$).
Proof. We provide the proof for which case $I$ is an ideal in $C_c(X)$. In the same way, the proof holds for the ideal $I$ in $C_c^*(X)$. (i) Clearly, the result holds if $I = C_c(X)$. Suppose that $I \varsubsetneq C_c(X)$. Let $f, g \in \text{cl}I$, $h \in C_c(X)$ and $u$ be a positive unit of $C_c(X)$. Then for some $f' \in B(f, \frac{u}{2}) \cap I$, and $g' \in B(g, \frac{u}{2}) \cap I$, we have $f' + g' \in B(f + g, u) \cap I$. To show that $fh \in \text{cl}I$, we consider the positive unit

$$u_1 = \frac{u}{(|h| + 1)(u + 1)} \in C_c(X).$$

Therefore, for some $f_1 \in B(f, u_1) \cap I$ we have that $|fh - f_1h| < u_1|h| < u$. So $f_1h \in B(fh, u) \cap I$. Moreover, if $f \in \text{cl}I$, then also $-f \in \text{cl}I$. Thus, $\text{cl}I$ contains both $f + g$ and $fh$. So $\text{cl}I$ is ideal. (ii) Suppose that $I$ is a proper ideal in $C_c(X)$ and $\text{cl}I = C_c(X)$. Consider the constant function $1 \in \text{cl}I$ and $0 < \varepsilon < 1$. Hence, the nonempty set $B(1, \varepsilon) \cap I$ contains a nonzero element of $C_c(X)$, $f$ say. Since $1 - \varepsilon < f(x) < 1 + \varepsilon$ for each $x \in X$, we have $Z(f) = \emptyset$, i.e., $f$ is a unit of $C_c(X)$, which is impossible (because $f \in I$). Thus, $\text{cl}I \subsetneq C_c(X)$, and we are done. 

The next result is now immediate.

Corollary 2.2. Any maximal ideal in $C_c(X)$ (resp. $C_c^*(X)$) and hence any intersection of maximal ideals in $C_c(X)$ (resp. $C_c^*(X)$) is closed.

Definition 2.3. An ideal $I$ in a commutative ring with unity $R$ is called a $z$-ideal in $R$ if for each $a \in I$, we have $M_a \subseteq I$, here $M_a$ is the intersection of all maximal ideals in $R$ containing $a$.

Evidently, each maximal ideal in $R$ is a $z$-ideal. This notion of $z$-ideal is consistent with the notion of $z$-ideals in $C(X)$, see [5, 4A(5)].

Proposition 2.4. Let $X$ be zero-dimensional and $I$ be an ideal in $C_c^*(X)$. Then $I$ is a $z$-ideal if and only if $g \in I$ whenever $Z(f^\beta) \subseteq Z(g^\beta)$ with $f \in I$ and $g \in C_c^*(X)$, where $f^\beta$ is the extension of $f$ to $\beta X$.

Proof. ($\Rightarrow$): Let $f \in I$, $g \in C_c^*(X)$ and $Z(f^\beta) \subseteq Z(g^\beta)$ and let $M_f$ be the intersection of all the maximal ideals in $C_c^*(X)$ containing $f$. By the assumption, $M_f \subseteq I$. Let $M$ be a maximal ideal in $C_c^*(X)$ containing $f$. According to [9, Corollary 2.11], $M$ has a form of $M^*_p = \{h \in C_c^*(X) : h^\beta(p) = 0\}$, for some $p \in \beta X$. Now, $Z(f^\beta) \subseteq Z(g^\beta)$ implies that $g \in M$. Hence, $g \in I$.

($\Leftarrow$): Let $f \in I$ and $g \in M_f$. Then $f \in M^*_p$ implies that $g \in M^*_p$, i.e., $Z(f^\beta) \subseteq Z(g^\beta)$. Therefore, by the hypothesis, $g \in I$.

Lemma 2.5. Let $X$ be zero-dimensional and $I$ be an ideal in $C_c(X)$. Then $I$ is a $z$-ideal if and only if it is a $z_c$-ideal.

Proof. ($\Rightarrow$): Let $I$ be a $z$-ideal in $C_c(X)$, $f \in I$ and $Z(f) \subseteq Z(g)$ with $g \in C_c(X)$. We have to show that $g \in I$. Since $I$ is a $z$-ideal, we have $M_f \subseteq I$, where $M_f$ is the intersection of all the maximal ideals in $C_c(X)$ containing $f$. It suffices to show that $g \in M_f$. So let $M^*_p (p \in \beta_0 X)$ be any maximal ideal in $C_c(X)$ which contains $f$, we have to show that $g \in M^*_p$ (see [1, Theorem
4.2]). Indeed \( f \in M_p \) implies that \( p \in \text{cl}_{\beta_0}XZ(f) \) which further implies that 
\( p \in \text{cl}_{\beta_0}XZ(g) \), by the assumption, \( Z(f) \subseteq Z(g) \). Hence, \( g \in M_p \). Thus, \( I \) becomes a \( z_c \)-ideal in \( C_c(X) \).

\((\Leftarrow)\) : Let \( I \) be a \( z_c \)-ideal in \( C_c(X) \) and \( f \in I \). We must show \( M_f \subseteq I \). Let \( g \in M_f \). Then \( f \in M_p \) gives \( g \in M_p \), where \( p \in \beta_0X \). Equivalently, 
\( \text{cl}_{\beta_0}XZ(f) \subseteq \text{cl}_{\beta_0}XZ(g) \). So
\( Z(f) = \text{cl}_{\beta_0}XZ(f) \cap X \subseteq \text{cl}_{\beta_0}XZ(g) \cap X = Z(g) \).

Now, the assumption yields that \( g \in I \).

Proposition 2.6. If \( I \) is a closed ideal in \( C_c(X) \), then \( I \) is a \( z_c \)-ideal.

Proof. Suppose that \( Z(f) \subseteq Z(g) \), \( f \in I \) and \( g \in C_c(X) \). To show that \( g \in I \), we show that \( g \in \text{cl}I \) because \( I = \text{cl}I \). Let \( u \in C_c(X) \) be a positive unit and let us define a function \( h : X \rightarrow \mathbb{R} \) as follows:

\[
\begin{align*}
    h(x) &= \begin{cases} 
        g(x) - \frac{u(x)}{f(x)} & \text{where } g(x) \geq \frac{u(x)}{2}, \\
        0 & \text{where } |g(x)| \leq \frac{u(x)}{2}, \\
        g(x) + \frac{u(x)}{f(x)} & \text{where } g(x) \leq -\frac{u(x)}{2}.
    \end{cases}
\end{align*}
\]

From the continuity of \( h \) on the three closed sets \( (g - \frac{u}{2})^{-1}([0, \infty]) \), \( (g + \frac{u}{2})^{-1}([0, \infty]) \cap (g - \frac{u}{2})^{-1}((\infty, 0]) \), and \( (g + \frac{u}{2})^{-1}((\infty, 0]) \), which their union is \( X \), we infer that \( h \in C(X) \). Moreover, since the ranges of \( g \) and \( f \) are countable, the range of \( h \) is also countable, i.e., \( h \in C_c(X) \). Thus, \( fh \in I \). Furthermore, it is easy to see that \( |g(x) - f(x)h(x)| < u(x) \) for every \( x \in X \), i.e., \( fh \in B(g, u) \cap I \) and thus \( g \in \text{cl}I \), which completes the proof.

The next example shows that the converse of the above proposition is not true in general.

Example 2.7. Consider the zero-dimensional space \( X = \mathbb{Q} \times \mathbb{Q}, \) \( p = (0, 0) \in X \), and put \( O_p = \{ f \in C(X) : p \in \text{int}_X Z(f) \} \) (note, \( C_c(X) = C(X) \) because \( X \) is countable). Recall that \( O_p \) is a \( z_c \)-ideal. We now claim that \( O_p \) is not a closed ideal in \( C(X) \). To see this, consider \( f(x, y) = \frac{|x|}{1 + |x| + |y|} \in C(X) \) and let \( u \) be a fixed positive unit of \( C(X) \). Define a function \( g \) by

\[
    g(x, y) = \begin{cases} 
        0 & \text{where } f(x, y) \leq \frac{u(x, y)}{2}, \\
        f(x, y) - \frac{u(x, y)}{2} & \text{where } f(x, y) \geq \frac{u(x, y)}{2}.
    \end{cases}
\]

Obviously, \( g \in C(X) \). Let \( G = \{(x, y) \in X : f(x, y) < \frac{u(x, y)}{2}\} \). Then \( p \in G \subseteq Z(g) \) and therefore \( g \in O_p \), in fact, \( g \in B(f, u) \cap O_p \). It follows that \( f \in \text{cl}_{C(X)}O_p \). On the other hand, the set \( Z(f) = \{p\} \) is not open in \( X \). Hence, \( f \in \text{cl}_{C(X)}O_p \setminus O_p \), i.e., \( O_p \) is not a closed ideal in \( C(X) \).

A Banach algebra \( B \) is an algebra that is a Banach space with a norm that satisfies \( \|xy\| \leq \|x\|\|y\| \) for all \( x, y \in B \), and there exists a unit element \( e \in B \) such that \( ex = xe = x, \|e\| = 1 \).

In [7, Definition 2.2], a topological space \( X \) is called a countably uniform closed-space, briefly, a CUC-space, if whenever \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of functions of \( C_c(X) \) and \( f_n \rightarrow f \) uniformly, then \( f \) belongs to \( C_c(X) \).
Theorem 2.8. If $X$ is a CUC-space, then $C_c^*(X)$ with the supremum-norm topology is a Banach algebra.

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence of functions in $C_c^*(X)$. Given $\varepsilon > 0$, we can find a natural number $N$ such that $\|f_n - f_m\| \leq \varepsilon$ for every $m, n > N$. Thus, $|f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in X$ and all $m, n > N$. Let $x \in X$ be fixed and $a_x$ be the limit of the numerical sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ (note, $\mathbb{R}$ is a Banach space). Now, define $f : X \to \mathbb{R}$ by $f(x) = a_x$. Let $n$ be fixed, then $|f_n(x) - \lim_{m \to \infty} f_m(x)| \leq \varepsilon$ for each $x \in X$ and each $m > N$. So $\|f_n - f\| \leq \varepsilon$. Since $n$ is arbitrary, we get $f_n \to f$ in the norm, uniformly. Consequently, $f \in C(X)$. Furthermore, our assumption implies that $f \in C_c(X)$. Moreover, $\|f\| \leq \|f - f_n\| + \|f_n\|$ gives $f$ is bounded. Hence, $C_c^*(X)$ is a Banach space. The proof is completed by the fact that $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in C_c^*(X)$.

3. Closed ideals in $C_c(X)$ and $C_c^*(X)$ (with the $m_c$-topology)

We need the next statement which is the counterpart of [5, 1D(1)] for $C_c(X)$.

Proposition 3.1. If $f, g \in C_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in C_c(X)$.

Proposition 3.2. Let $X$ be a zero-dimensional space, $f \in C_c(\beta_0 X)$ and let $f_0$ be the restriction of $f$ on $X$. Then $\text{int}_{\beta_0 X} Z(f) \subseteq \text{cl}_{\beta_0 X} Z(f_0) \subseteq Z(f)$.

Proof. Let $p \in \text{int}_{\beta_0 X} Z(f)$ and $V$ be an open set in $\beta_0 X$ containing $p$. Since $X$ is dense in $\beta_0 X$, we have $\emptyset \neq V \cap \text{int}_{\beta_0 X} Z(f) \cap X \subseteq V \cap Z(f_0)$. So $p \in \text{cl}_{\beta_0 X} Z(f_0)$. For the second inclusion, since $Z(f_0) \subseteq Z(f)$, we have that $\text{cl}_{\beta_0 X} Z(f_0) \subseteq \text{cl}_{\beta_0 X} Z(f) = Z(f)$.

Corollary 3.3. Let $X$ be zero-dimensional and $p \in \beta_0 X$. Then

(i) $\bigcap_{f \in M_p} \text{cl}_{\beta_0 X} Z(f) = \{p\}$.

(ii) If $p \in X$, then $\bigcap_{f \in M_{cp}} Z(f) = \{p\}$, i.e., $M_{cp}$ is fixed.

Proof. (i). Recall that $f \in M_p^c$ if and only if $p \in \text{cl}_{\beta_0 X} Z(f)$ (see [1, Theorem 4.2]). Therefore, $p \in \bigcap_{f \in M_p^c} \text{cl}_{\beta_0 X} Z(f)$. Now, we claim that the latter intersection is the singleton set $\{p\}$. On the contrary, suppose that this set contains an element $q \in \beta_0 X$ distinct from $p$. Since $\beta_0 X$ is zero-dimensional, by [3, Proposition 4.4], there exists $g \in C_c(\beta_0 X)$ such that $p \in \text{int}_{\beta_0 X} Z(g)$ and $g(q) = 1$. Let $g_0$ be the restriction of $g$ on $X$. Then by Proposition 3.2, $\text{cl}_{\beta_0 X} Z(g_0)$ contains $p$ but not $q$. This means that $g_0 \in M_p^c \setminus M_p^g$ which is a contradiction, so (i) holds. (ii). Clearly, $\bigcap_{f \in M_{cp}} Z(f) = \bigcap_{f \in M_{cp}} \text{cl}_{\beta_0 X} Z(f) \cap X = \{p\}$.

In a similar way to Proposition 3.2 and Corollary 3.3, we get:

Proposition 3.4. For a Tychonoff space $X$ and $f \in C^*(X)$, we have that $\text{int}_{\beta X} Z(f^\beta) \subseteq \text{cl}_{\beta X} Z(f) \subseteq Z(f^\beta)$, where $f^\beta$ is the extension of $f$ to $\beta X$. Moreover, if $p \in \beta X$, then $\bigcap_{f \in M_p} \text{cl}_{\beta X} Z(f) = \{p\}$. In particular, if $p \in X$, then $\bigcap_{f \in M_p} Z(f) = \{p\}$, i.e., $M_p$ is fixed.
Proposition 3.5. Let $X$ be zero-dimensional, $p \in \beta_0X$ and $\pi_p$ be its corresponding point of $\beta X$ in characterizing of maximal ideals in $C_c(X)$. Then $M_p^p \cap C_c^*(X) \subseteq M^{*p} \cap C_c^*(X)$. Particularly, if $X$ is strongly zero-dimensional, then $M_p^p \cap C_c^*(X) \subseteq M^{*p} \cap C_c^*(X)$.

Proof. In view of [1, Theorems 4.2, 4.8], we have

$$M_p^p = \{ f \in C_c(X) : p \in \text{cl}_{\beta_0X} Z(f) \} = \{ f \in C_c(X) : \pi_p \in \text{cl}_{\beta X} Z(f) \}.$$ Let $f \in M_p^p \cap C_c^*(X)$. Then $\pi_p \in \text{cl}_{\beta X} Z(f)$ and hence $f^\beta(\pi_p) = 0$, by Proposition 3.4. Therefore, $f \in M^{*p} \cap C_c^*(X)$. The second part follows from the assumption, i.e., $\beta_0X = \beta X$ and so $\pi_p = p$. \qed

Remark 3.6. Replacing $T$ with $\beta_0X$ in [1, Proposition 3.2] implies that for any two zero-sets $Z_1$ and $Z_2$ in $Z_c(X)$, we get $\text{cl}_{\beta_0X} (Z_1 \cap Z_2) = \text{cl}_{\beta_0X} Z_1 \cap \text{cl}_{\beta_0X} Z_2$.

Remark 3.7. ([1, Remark 4.12]) If $X$ is zero-dimensional and $f, g \in C_c(X)$, then $\text{cl}_{\beta_0X} Z(f)$ is a neighborhood of $\text{cl}_{\beta_0X} Z(g)$ if and only if there exists $h \in C_c(X)$ such that $Z(g) \subseteq \text{coz}(h) \subseteq Z(f)$.

Proposition 3.8. Let $X$ be zero-dimensional and $I$ a proper ideal in $C_c(X)$ and let $V_c(I) = \{ p \in \beta_0X : M_p^p \supseteq I \}$. Then:

(i) $V_c(I) = \bigcap_{g \in I} \text{cl}_{\beta_0X} Z(g)$.
(ii) If $f \in C_c(X)$ and $\text{cl}_{\beta_0X} Z(f)$ is a neighborhood of $V_c(I)$, then $f \in I$.

Proof. (i). This is easily obtained from the fact that $g \in M_p^p$ if and only if $p \in \text{cl}_{\beta_0X} Z(g)$. (ii). Suppose that

$$V_c(I) = \bigcap_{g \in I} \text{cl}_{\beta_0X} Z(g) \subseteq \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f).$$

Then we have $\bigcup_{g \in I} (\beta_0X \setminus \text{cl}_{\beta_0X} Z(g)) \supseteq \beta_0X \setminus \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f)$. Hence, the collection

$$C = \{ \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f), \beta_0X \setminus \text{cl}_{\beta_0X} Z(g) : g \in I \}$$

is an open cover for the compact set $\beta_0X$. Therefore, there is a finite number of elements of $I$: $g_1, g_2, \ldots, g_n$ say, such that

$$\beta_0X = \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f) \cup (\beta_0X \setminus \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f))$$

$$= \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f) \cup \left( \bigcup_{i=1}^n (\beta_0X \setminus \text{cl}_{\beta_0X} Z(g_i)) \right).$$

Now, we have that

$$\left( \bigcap_{i=1}^n \text{cl}_{\beta_0X} Z(g_i) \right) \cap (\beta_0X \setminus \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f)) = \varnothing.$$

Thus, $\bigcap_{i=1}^n \text{cl}_{\beta_0X} Z(g_i) \subseteq \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f)$. Since $I$ is a proper ideal, the element $g = \sum_{i=1}^n g_i^2$ of $I$ is not a unit of $C_c(X)$ and hence $Z(g) = \bigcap_{i=1}^n Z(g_i) \neq \varnothing$. From Remark 3.6 we conclude that

$$\text{cl}_{\beta_0X} Z(g) = \text{cl}_{\beta_0X} \left( \bigcap_{i=1}^n Z(g_i) \right) = \bigcap_{i=1}^n \text{cl}_{\beta_0X} Z(g_i) \subseteq \text{int}_{\beta_0X} \text{cl}_{\beta_0X} Z(f).$$

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This leads us \( \text{cl}_{\beta_0} X Z(f) \) is a neighborhood of \( \text{cl}_{\beta_0} X Z(g) \). In view of Remark 3.7, there exists \( h \in C_c(X) \) such that \( Z(g) \subseteq \text{coz}(h) \subseteq Z(f) \). So \( Z(f) \) is a neighborhood of \( Z(g) \). By Proposition 3.1, we get \( f \in I \). \( \square \)

**Lemma 3.9.** Let \( X \) be zero-dimensional and \( g \in C_c(X) \). Then for any neighborhood \( B(g, u) \) of \( g \) in the \( m_c \)-topology, there exists some \( f_u \in B(g, u) \) such that \( \text{cl}_{\beta_0} X Z(f_u) \) is a neighborhood of \( \text{cl}_{\beta_0} X Z(g) \).

**Proof.** If \( \text{cl}_{\beta_0} X Z(g) \) is an open set in \( \beta_0 X \), then we set \( f_u = g \). In general, we define a function \( f_u : X \to \mathbb{R} \) by

\[
  f_u(x) = \begin{cases} 
  g(x) - \frac{u(x)}{2} & \text{where } g(x) \geq \frac{u(x)}{2}, \\
  0 & \text{where } |g(x)| \leq \frac{u(x)}{2}, \\
  g(x) + \frac{u(x)}{2} & \text{where } g(x) \leq -\frac{u(x)}{2}.
  \end{cases}
\]

It is clear that \( f_u \in C(X) \) and further since the range of \( g \) and \( u \) is countable, we get \( f_u \in C_c(X) \). Moreover, \( f_u \in B(g, u) \). To establish the conclusion, consider the function \( h \) below

\[
  h(x) = \begin{cases} 
  (g(x) + \frac{u(x)}{2})(g(x) - \frac{u(x)}{2}) & \text{where } |g(x)| \leq \frac{u(x)}{2}, \\
  0 & \text{where } |g(x)| \geq \frac{u(x)}{2}.
  \end{cases}
\]

We observe that \( h \in C_c(X) \). Furthermore, \( Z(g) \subseteq \text{coz}(h) \subseteq Z(f_u) \). Now, Remark 3.7 implies that \( \text{cl}_{\beta_0} X Z(f_u) \) is a neighborhood of \( \text{cl}_{\beta_0} X Z(g) \), and we are through. \( \square \)

**Theorem 3.10.** Let \( X \) be zero-dimensional and \( I \) a proper ideal in \( C_c(X) \) and let \( V_c(I) \) be the same as the set in Proposition 3.8 \( (V_c(I) = \bigcap_{g \in I} \text{cl}_{\beta_0} X Z(g)) \).

Let

\[
  J = \{ f \in C_c(X) : \text{cl}_{\beta_0} X Z(f) \supseteq V_c(I) \}, \text{ and } \bar{I} = \cap \{ M^p_c : M^p_c \supseteq I \}.
\]

Then:

(i) \( \bar{I} \) is a closed ideal in \( C_c(X) \) containing \( I \).
(ii) \( J = \bar{I} \), in other words, \( J \) is the kernel of the hull of \( I \) in the structure space of \( C_c(X) \).
(iii) \( V_c(I) = V_c(\bar{I}) \).
(iv) \( \text{cl} I = \bar{I} \).

**Proof.** (i). It follows from Corollary 2.2. (ii). Let \( f \in J \) and \( M^p_c (p \in \beta_0 X) \) be a maximal ideal in \( C_c(X) \) containing \( I \). Then

\[
  (3.1) \quad V_c(I) \supseteq V_c(M^p_c) \text{ and so } \text{cl}_{\beta_0} X Z(f) \supseteq V_c(I) \supseteq V_c(M^p_c) = \{ p \}
\]

(note, the last equality follows from Corollary 3.3). Therefore, \( f \in M^p_c \) and thus \( f \in \bar{I} \), i.e., \( J \subseteq \bar{I} \). For the reverse inclusion, we show that if \( f \notin J \), then \( f \notin \bar{I} \). Since \( f \notin J \), there exists \( q \in \beta_0 X \) such that \( q \in V_c(I) \setminus \text{cl}_{\beta_0} X Z(f) \). Therefore, \( g \in M^q_f \) for every \( g \in I \) and hence \( I \subseteq M^q_f \). But \( f \notin M^q_f \). Thus, \( M^q_f \) is a maximal ideal containing \( I \) but not \( f \). This yields that \( f \notin \bar{I} \). (iii). Using (ii) and the definition of \( J \), we have \( V_c(\bar{I}) = V_c(J) \supseteq V_c(I) \). On the other hand, the inclusion \( I \subseteq \bar{I} \) implies that \( V_c(\bar{I}) \subseteq V_c(I) \). So (iii) holds.
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(iv). By (i), $\text{cl}I \subseteq \bar{I}$. Now, suppose that $g \in \bar{I}$ and $u$ is a positive unit of $C_c(X)$. We claim that $B(g,u) \cap I \neq \emptyset$. According to Lemma 3.9, there exists $f_u \in C_c(X)$ such that $f_u \in B(g,u)$, and $\text{cl}_{\beta_X} \beta(X(f_u))$ is a neighborhood of $\text{cl}_{\beta_X} \beta(X(g))$. Now, it remains to show that $f_u \in I$. From (iii), we infer that $V_c(I) = V_c(\bar{I}) \subseteq \text{cl}_{\beta_X} \beta(X(g)) \subseteq \text{int}_{\beta_X} \beta(X(f_u))$. Proposition 3.8(ii) now yields that $f_u \in I$. Therefore, $f_u \in B(g,u) \cap I$ and so $g \in \text{cl}I$, i.e., $\bar{I} \subseteq \text{cl}I$. □

It is known that a proper ideal in $C(X)$ with the $m$-topology is closed if and only if it is an intersection of maximal ideals in $C(X)$ (see [5, 7Q(2)]). The next theorem involves the countable analogue characterization of closed ideals in $C_c(X)$. Using Theorem 3.10(iv) and Corollary 2.2, we obtain:

**Theorem 3.11.** Let $X$ be zero-dimensional and the topology on $C_c(X)$ be the $m_c$-topology. Then a proper ideal in $C_c(X)$ is closed if and only if it is an intersection of maximal ideals of $C_c(X)$.

**Theorem 3.12.** Let $X$ be zero-dimensional and the topology on $C_c(X)$ (resp. $C(X)$) be the $m_c$-topology (resp. the $m$-topology). Then the following statements are equivalent.

(i) Every ideal in $C(X)$ is closed.

(ii) $X$ is a $P$-space.

(iii) Every ideal in $C_c(X)$ is closed.

(iv) Every prime ideal in $C_c(X)$ is closed.

**Proof.** (i) $\Rightarrow$ (ii). It follows from [5, 4J(9), 7Q(2)].

(ii) $\Rightarrow$ (iii). By [3, Proposition 5.3], $X$ is a $CP$-space. Now, the result is obtained by [3, Theorem 5.8(7)] and Corollary 2.2.

(iii) $\Rightarrow$ (iv). It is evident.

(iv) $\Rightarrow$ (ii). According to [3, Corollary 5.7], it is enough to show that $X$ is a $CP$-space. Let $P$ be a prime ideal in $C_c(X)$, then by [1, Lemma 4.11(4)], $P$ is contained in a unique maximal ideal $M^p$ of $C_c(X)$, where $p \in \beta X$. Now, by the assumption and Theorem 3.11, we get $P = M^p$, i.e., $X$ is a $CP$-space. □

**Theorem 3.13.** Let $X$ be strongly zero-dimensional and the topology on $C^*_c(X)$ (resp. $C^*(X)$) be the $m_c$-topology (resp. the $m$-topology). Then the following statements are equivalent.

(i) Every properly closed ideal in $C^*_c(X)$ is an intersection of maximal ideals of $C^*_c(X)$.

(ii) $X$ is pseudocompact.

(iii) Every properly closed ideal in $C^*(X)$ is an intersection of maximal ideals of $C^*(X)$.

**Proof.** A maximal ideal in $C^*_c(X)$ is of the form $M^p_{\alpha} = \{f \in C^*_c(X) : f^\alpha(p) = 0\}$, where $p \in \beta X$. Also, $M^p = M_{\alpha} \cap C^*_c(X)$, see [9, Corollaries 2.10, 2.11].

(i) $\Rightarrow$ (ii). Suppose that $X$ is not pseudocompact, so $C^*_c(X) \varsubsetneq C_c(X)$, by [9, Theorem 6.3]. Hence, $C_c(X)$ contains an unbounded element, $f$ say. So for some $p \in \beta X$ and the maximal ideal $M^p$ of $C_c(X)$, we have $|M^p(f)|$ is infinitely
large ([9, Proposition 2.4]). In other words, $M^p_\varepsilon$ is hyper-real, i.e., $\mathbb{R} \subsetneq \frac{C_\varepsilon(X)}{M^p_\varepsilon}$.

Hence, by [9, Corollary 2.13], $M^p_\varepsilon \cap C_\varepsilon^*(X)$ is not a maximal ideal in $C_\varepsilon^*(X)$. Using Proposition 3.5, we infer that

\begin{equation}
M^p_\varepsilon \cap C_\varepsilon^*(X) \subsetneq M^{*p} \cap C_\varepsilon^*(X). 
\end{equation}

Furthermore, since the maximal ideal $M^p_\varepsilon$ is closed in $C_\varepsilon(X)$ (Corollary 2.2), the ideal $M^p_\varepsilon \cap C_\varepsilon^*(X)$ is also closed in $C_\varepsilon^*(X)$. We now claim that the latter closed ideal cannot be an intersection of maximal ideals of $C_\varepsilon^*(X)$. Otherwise,

\begin{equation}
M^p_\varepsilon \cap C_\varepsilon^*(X) = \bigcap_{q \in A \subseteq \beta X} \left( M^{*q} \cap C_\varepsilon^*(X) \right),
\end{equation}

for a subset $A$ of $\beta X$. Notice that by (3.2), $A \neq \emptyset$ since $p \in A$. Now, we claim that $A = \{p\}$. On the contrary, suppose that $A$ contains an element $q$ distinct from $p$. We can take $f \in C_\varepsilon(\beta X)$ such that $Z(f)$ is a neighborhood of $p$ and $f(q) = 1$ (note, by the assumption, $\beta X$ is zero-dimensional). Let $f_0$ be the restriction of $f$ on $X$. Then the compactness of $\beta X$ gives $f$ and hence $f_0$ are bounded, i.e., $f_0 \in C_\varepsilon(X)$. By density of $X$ in $\beta X$, we get $f = f_0^\beta$, where $f_0^\beta$ is the extension of $f_0$ to $\beta X$. Due to Proposition 3.2, we infer that $p \in \text{cl}_{\beta X} Z(f_0)$, since $p \in \text{int}_{\beta X} Z(f)$. Hence, $f_0 \in M^p_\varepsilon \cap C_\varepsilon^*(X)$.

On the other hand, since $q \notin Z(f)$, we have that $f_0 \notin M^{*q}$. Therefore, $f_0 \in M^p_\varepsilon \cap C_\varepsilon^*(X) \setminus (M^{*q} \cap C_\varepsilon^*(X))$, which contradicts the equation in (3.3). So $A = \{p\}$ and hence $M^p_\varepsilon \cap C_\varepsilon^*(X) = M^{*p} \cap C_\varepsilon^*(X)$. But this also contradicts (3.2). Thus, if $X$ is not pseudocompact, then there exists a closed ideal in $C_\varepsilon^*(X)$ which is not an intersection of maximal ideals of $C_\varepsilon^*(X)$, and we are done.

(ii) $\Rightarrow$ (i). Since $X$ is pseudocompact, $C(X) = C^*(X)$ gives $C_\varepsilon(X) = C_\varepsilon^*(X)$. Now, it follows from Theorem 3.11.

(ii) $\Leftrightarrow$ (iii). It follows from [5, 7Q(3)].

We end the article with some results on $e_\varepsilon$-filters on $X$ and $e_\varepsilon$-ideals in $C_\varepsilon^*(X)$, for more details, see [14, Section 2]. Let $p \in \beta X$ and $f^\beta$ be the extension of $f \in C^*(X)$ to $\beta X$. Let us recall that

\begin{equation}
M^{*p} = \{ f \in C^*_\varepsilon(X) : f^\beta(p) = 0 \} = M^{*p} \cap C^*_\varepsilon(X), \quad \text{and} \quad O^{*p}_\varepsilon = O^p_\varepsilon \cap C^*_\varepsilon(X),
\end{equation}

where

\begin{equation}
M^{*p} = \{ f \in C^*(X) : f^\beta(p) = 0 \}, \quad \text{and} \quad O^p_\varepsilon = \{ f \in C_\varepsilon(X) : p \in \text{int}_{\beta X} \text{cl}_{\beta X} Z(f) \}.
\end{equation}

**Lemma 3.14.** Let $X$ be strongly zero-dimensional and $p \in \beta X$. Then

\begin{equation}
E_\varepsilon(M^{*p}_\varepsilon) = Z_\varepsilon(O^p_\varepsilon) = Z_\varepsilon(O^{*p}_\varepsilon) = E_\varepsilon(O^{*p}_\varepsilon).
\end{equation}

**Proof.** By the hypothesis, $\beta X = \beta_0 X$. To get the result, we show the following chain of inclusions holds.

\begin{equation}
E_\varepsilon(M^{*p}_\varepsilon) \subseteq Z_\varepsilon(O^p_\varepsilon) \subseteq Z_\varepsilon(O^{*p}_\varepsilon) \subseteq E_\varepsilon(O^{*p}_\varepsilon) \subseteq E_\varepsilon(M^{*p}_\varepsilon).
\end{equation}

To establish the first inclusion, let $E_\varepsilon(f) := \{ x \in X : |f(x)| \leq \varepsilon \} \in E_\varepsilon(M^{*p}_\varepsilon)$, where $f \in M^{*p}_\varepsilon$ and $\varepsilon > 0$. Then $f^\beta(p) = 0$. Notice that $E_\varepsilon(f) = Z((|f| - \varepsilon)\vee 0)$.
Since \( e \subset C \), in other words, \( \{ (f \mid - \varepsilon) \mid 0 \in O^p_\varepsilon \} \), we have

(3.5) \( \text{cl}_{\beta X} Z((f \mid - \varepsilon) \cup 0) = \text{cl}_{\beta X} E_c^\varepsilon(f) = \{ q \in \beta X : |f\beta(q)| \leq \varepsilon \}. \)

Hence, \( p \in \text{int}_{\beta X} \text{cl}_{\beta X} Z((f \mid - \varepsilon) \cup 0) \), we have \( \text{cl}_{\beta X} E_c^\varepsilon(f) = \{ q \in \beta X : |f\beta(q)| \leq \varepsilon \}. \)

Here, we are going to show the last equality in (3.5). Let \( q \in \beta X \) such that \( |f\beta(q)| \leq \varepsilon \). Since \( E \) is dense in \( \beta X \), there exists a net \( (x_\lambda)_{\lambda \in \Lambda} \subseteq X \) converging to \( q \) and so \( f(x_\lambda) = f\beta(x_\lambda) \rightarrow f\beta(q) \). Moreover, \( |f(x_\lambda)| \rightarrow |f\beta(q)| \). Now, let \( V \) be an open set in \( \beta X \) containing \( q \). Then for some \( \lambda_0 \in \Lambda \) and each \( \lambda \geq \lambda_0 \), we have \( x_\lambda \in V \). Furthermore, \( |f\beta(q)| \leq \varepsilon \) yields that \( |f(x_\lambda)| \leq \varepsilon \). Hence, \( V \cap E_c^\varepsilon(f) \neq \emptyset \), i.e., \( q \in \text{cl}_{\beta X} E_c^\varepsilon(f) \).

The second inclusion in (3.4) follows from the fact that \( Z(f) = Z(f, p) \), where \( f \in O^p_\varepsilon \) (and thus \( f \in O^p_\varepsilon \)). To verify the third inclusion, we let \( f \in O^p_\varepsilon \) and show that \( Z(f) \in E_c(O^p_\varepsilon) \). Since \( p \) does not belong to the closed set \( F := \beta X \setminus \text{int}_{\beta X} \text{cl}_{\beta X} Z(f) \) and \( \beta X \) is zero-dimensional, by [3, Proposition 4.4], there is some \( g \in C_\epsilon(\beta X) = C_\epsilon^\varepsilon(\beta X) \) such that \( p \in \text{int}_{\beta X} Z(g) \) and \( g(F) = \{1\} \). Let \( g_0 \) be the restriction of \( g \) on \( X \). Then by Proposition 3.2, \( p \in \text{int}_{\beta X} Z(g_0) \). So \( g_0 \in O^p_\varepsilon \) and hence \( E_c^\varepsilon(g_0) \in E_c(O^p_\varepsilon) \) for all \( \varepsilon > 0 \). Let \( 0 < \varepsilon < 1 \) be fixed. Since \( E \) is dense in \( \beta X \), the open set \( \{ q \in \beta X : |g(q)| < \varepsilon \} \) intersects \( X \) nontrivially (since it contains \( p \)). Therefore, \( \emptyset \neq \{ q \in \beta X : |g(q)| \leq \varepsilon \} \) and \( \text{cl}_{\beta X} E_c^\varepsilon(g_0) \subseteq (\beta X \setminus F) \cap X \subseteq Z(f) \).

Now, since the \( Z_c \)-filter (in fact, the \( e_c \)-filter) \( E_c(O^p_\varepsilon) \) contains \( E_c^\varepsilon(g_0) \) and \( E_c^\varepsilon(g_0) \subseteq Z(f) \), we infer that \( Z(f) \in E_c(O^p_\varepsilon) \), and we are done. Finally, the last inclusion in (3.4) follows from the inclusion \( O^p_\varepsilon \subseteq M^p \) and the fact that \( E_c \) preserves the order, see [14, Corollary 2.1].

**Theorem 3.15.** Let \( X \) be a \( P \)-space and \( F \), an \( e_c \)-filter on \( X \). Then \( F \) is an \( e_c \)-ultrafilter if and only if it is a \( z_c \)-ultrafilter.

**Proof.** (\( \Rightarrow \)) By [5, 4K(7), 6M(1), 16O], every \( P \)-space is strongly zero-dimensional (see also [15, Proposition 2.12]). By [5, 7L], we have \( O^p = M^p \) for every \( p \in \beta X \). Therefore, \( O^p_\varepsilon = O^p \cap C_\epsilon(X) = M^p \cap C_\epsilon(X) = M^p \) (note, \( \beta X = \beta_0 X \)). Let \( F \) be an \( e_c \)-ultrafilter on \( X \). Then \( E_c^{-1}(F) \) is a maximal ideal in \( C_\epsilon^\varepsilon(X) \), see [14, Proposition 2.14]. Therefore, \( E_c^{-1}(F) = M^p \) for some \( p \in \beta X \). By Lemma 3.14, we have

\[
F = E_c(E_c^{-1}(F)) = E_c(M^p) = Z_c[M^p] = Z_c[M^p]
\]

Since \( M^p \) is a maximal ideal in \( C_\epsilon(X) \), \( F \) is a \( z_c \)-ultrafilter.

(\( \Leftarrow \)) Suppose that \( F \) is a \( z_c \)-ultrafilter. Then \( Z_c^{-1}[F] \) is a maximal ideal in \( C_\epsilon(X) \). So \( Z_c^{-1}[F] = M^p \) for some \( p \in \beta X \). Therefore,

\[
F = Z_c[Z_c^{-1}[F]] = Z_c[M^p] = E_c(M^p)
\]

Since \( M^p \) is a maximal ideal in \( C_\epsilon^\varepsilon(X) \), \( F \) is an \( e_c \)-ultrafilter. \( \square \)

**Corollary 3.16.** For a strongly zero-dimensional space \( X \) and \( p \in \beta X \), \( M^p \) is the only \( e_c \)-ideal in \( C_\epsilon^\varepsilon(X) \) containing \( O^p_\varepsilon \).
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Proof. Let \( J \) be an \( e_c \)-ideal in \( C_c^*(X) \) which contains \( O_c^p \). Then 
\[
E_c^{-1}(E_c(O_c^p)) \subseteq E_c^{-1}(E_c(J)) = J.
\]
By Lemma 3.14, \( E_c(M_c^p) = E_c(O_c^p) \) and therefore 
\[
M_c^p = E_c^{-1}(E_c(M_c^p)) = E_c^{-1}(E_c(O_c^p)) \subseteq J.
\]
So \( M_c^p = J \), and we are through. \( \square \)

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