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From interpolative contractive mappings to generalized Ćirić-quasi contraction mappings

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Abstract

In this article we consider a restricted version of Ćirić-quasi contraction mapping for showing that this mapping generalizes several known interpolative type contractive mappings. Also here we introduce the concept of interpolative strictly contractive type mapping T and prove a fixed point theorem for such mapping over a T-orbitally compact metric space. Some examples are given in support of our established results. Finally we give an observation regarding (λ, α, β)-interpolative Kannan contractions introduced by Gaba et al.

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1. Introduction and preliminaries

In the year 1922, S. Banach had established a remarkable fixed point theorem, known as 'Banach Contraction Principle' which is given as follows:

Theorem 1.1 ([2]). *If a mapping T from a complete metric space (X, d) to itself satisfies the following condition*

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

for some $\alpha \in [0, 1)$ then T possesses a unique fixed point in X.

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Several generalizations of this theorem have been made by researchers, working in the area of fixed point theory, by means of different new type contractive mappings.

Recently E. Karapinar [7] proposed a new Kannan-type contractive mapping via the notion of interpolation and proved a fixed point theorem over metric space. In his paper, Karapinar assumed that the interpolative Kannan-type contractive mapping T over a metric space X satisfies the contractive condition for all $x, y \in X$ with $x \neq Tx$. But in this situation it is to be noted that if this mapping T has a fixed in X then it will be a constant mapping and therefore T has a unique fixed point trivially. To remove such triviality the authors in [8] assumed that interpolative type mappings satisfy the contractive condition for all $x, y \in X \setminus Fix(T)$, where $Fix(T)$ is the set of all fixed points of T. Though in this case an interpolative contractive type mapping may possesses more than one fixed point.

Definition 1.2 ([7])**.** In a metric space (X, d), a mapping $T : X \to X$ is said to be interpolative Kannan-type contractive mapping if it satisfies

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha} \text{ for all } x, y \in X \setminus Fix(T), \tag{1.2}$$

for some $\lambda \in [0, 1)$ and for some $\alpha \in (0, 1)$.

Theorem 1.3. [7] *Let (X, d) be a complete metric space and $T : X \to X$ be an interpolative Kannan-type contractive mapping. Then T has at least one fixed point in X.*

As an extension of interpolative Kannan-type contractive mappings, Karapinar et al. introduced interpolative Reich-Rus-Ćirić type contractions (See [8]). The definition is given below.

Definition 1.4 ([8])**.** In a metric space (X, d), a mapping $T : X \to X$ is called interpolative Reich-Rus-Ćirić type contraction mapping if it satisfies

$$d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta}[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha-\beta} \text{ for all } x, y \in X \setminus Fix(T), \tag{1.3}$$

for some $\lambda \in [0, 1)$ and for $\alpha, \beta \in (0, 1)$.

Theorem 1.5 ([8])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be an interpolative Reich-Rus-Ćirić type contraction mapping. Then T has a fixed point in X.*

Further extension of interpolative Kannan-type contractive mappings has been given by Karapinar et al. [9], which is known as interpolative Hardy-Rogers type contraction. The definition is given as follows.

Definition 1.6 ([9])**.** In a metric space (X, d), a mapping $T : X \to X$ is said to be interpolative Hardy-Rogers type contraction mapping if it satisfies

$$d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta}[d(x, Tx)]^{\alpha}[d(y, Ty)]^{\gamma}\left[\frac{1}{2}\left(d(x, Ty) + d(y, Tx)\right)\right]^{1-\alpha-\beta-\gamma} \tag{1.4}$$

for all $x, y \in X \setminus Fix(T)$, for some $\lambda \in [0,1)$ and for $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$.

Theorem 1.7 ([9]). *Let (X,d) be a complete metric space and $T : X \to X$ be an interpolative Hardy-Rogers type contraction mapping. Then T has at least one fixed point in X.*

Recently C. B. Ampadu [1] has defined interpolative Berinde weak operator in his paper. The definition is given as follows:

Definition 1.8 ([1]). Let (X,d) be a metric space. We say $T : X \to X$ is
(i) an interpolative Berinde weak operator if it satisfies

$$d(Tx,Ty) \le \lambda[d(x,y)]^{\alpha}[d(x,Tx)]^{1-\alpha} \text{ for all } x, y \in X \setminus Fix(T), \tag{1.5}$$

for some $\lambda \in [0,1)$ and for some $\alpha \in (0,1)$.
(ii) an alternate interpolative Berinde Weak operator if it satisfies

$$d(Tx,Ty) \le \lambda\sqrt{d(x,y)d(x,Tx)} \text{ for all } x, y \in X \setminus Fix(T), \tag{1.6}$$

where $\lambda \in [0,1)$.
Any interpolative Berinde weak operator is an alternate interpolative Berinde Weak operator.

Theorem 1.9 ([1]). *In a complete metric space (X,d) an interpolative Berinde weak operator T always possesses a fixed point.*

As a generalization of 'Banach Contraction Principle', Ćirić [3] had introduced a new contractive mapping known as Ćirić-quasi contraction mapping and proved a fixed point theorem for such mappings.

Theorem 1.10 ([3]). *Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping. If T satisfies the contractive condition*

$$d(Tx,Ty) \le k \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\} \textit{ for all } x, y \in X,$$

then T has a unique fixed point in X.

In the next section we find some new forms of interpolative contractive mappings and show that these interpolative contractive mappings are nothing but Ćirić-quasi contraction mappings.

2. Main results

Let (X,d) be a metric space, Δ_{IK} be the set of all interpolative Kannan type contractions on X and $\Delta_{SK} = \{T : X \to X : d(Tx,Ty) \le \lambda\sqrt{d(x,Tx)d(y,Ty)}$ for all $x, y \in X \setminus Fix(T)$, where $\lambda \in [0,1)\}$.

Theorem 2.1. *In a metric space (X,d), $\Delta_{IK} = \Delta_{SK}$.*

Proof. Clearly $\Delta_{SK} \subset \Delta_{IK}$. Now let $T \in \Delta_{IK}$ be chosen as arbitrary. Then there exists $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \lambda d(x,Tx)^{\alpha} d(y,Ty)^{1-\alpha} \text{ for all } x, y \in X \setminus Fix(T).$$

Now for any $x, y \in X \setminus Fix(T)$ we have

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha} \tag{2.1}$$

and also due to symmetry

$$d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y, Ty)^{\alpha} d(x, Tx)^{1-\alpha}. \tag{2.2}$$

Multiplying the inequalities (2.1) and (2.2) it follows that

$$d(Tx, Ty) \leq \lambda \sqrt{d(x, Tx) d(y, Ty)},$$

which proves that $T \in \Delta_{SK}$ and hence $\Delta_{IK} = \Delta_{SK}$. □

In a metric space (X, d), let Δ_{IR} be the set of all interpolative Reich-Rus-Ćirić type contractions on X and $\Delta_{SR} = \{T : X \to X : d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} \{d(x, Tx) d(y, Ty)\}^{\frac{1-\alpha}{2}}$ for all $x, y \in X \setminus Fix(T)$, where $\lambda \in [0, 1), \alpha \in (0, 1)\}$.

Theorem 2.2. *In a metric space (X, d), $\Delta_{IR} = \Delta_{SR}$.*

Proof. It is clearly seen that $\Delta_{SR} \subset \Delta_{IR}$. Now let $T \in \Delta_{IR}$ be chosen arbitrarily. Then there exists $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{\beta} d(y, Ty)^{1-\alpha-\beta} \text{ for all } x, y \in X \setminus Fix(T).$$

Now for any $x, y \in X \setminus Fix(T)$ we have

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{\beta} d(y, Ty)^{1-\alpha-\beta} \tag{2.3}$$

and also due to symmetry we get

$$d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y, x)^{\alpha} d(y, Ty)^{\beta} d(x, Tx)^{1-\alpha-\beta}. \tag{2.4}$$

Multiplying the inequalities (2.3) and (2.4) it follows that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} \{d(x, Tx) d(y, Ty)\}^{\frac{1-\alpha}{2}},$$

which proves that $T \in \Delta_{SR}$ and hence $\Delta_{IR} = \Delta_{SR}$. □

Remark 2.3. From the Theorem 2.2 we observe that, β has no importance to define interpolative Reich-Rus-Ćirić type contraction mappings.

Let us take Δ_{IH} as the set of all interpolative Hardy-Rogers type contractions and $\Delta_{SH} = \{T : X \to X : d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} \{d(x, Tx) d(y, Ty)\}^{\xi} \left(\frac{1}{2} [d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-2\xi}$ for all $x, y \in X \setminus Fix(T)$, where $\lambda \in [0, 1), \alpha, \xi \in (0, 1)$ such that $\alpha + 2\xi < 1\}$.

Theorem 2.4. *In a metric space (X, d), $\Delta_{IH} = \Delta_{SH}$.*

Proof. $\Delta_{SH} \subset \Delta_{IH}$ trivially. Now let $T \in \Delta_{IH}$ be taken as arbitrary. Then there exists $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{\beta} d(y, Ty)^{\gamma} \left(\frac{1}{2} [d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-\beta-\gamma}$$

for all $x, y \in X \setminus Fix(T)$. Now for any $x, y \in X \setminus Fix(T)$ we have

(2.5)
$$d(Tx,Ty) \leq \lambda d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} \left(\frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right)^{1-\alpha-\beta-\gamma}$$

and also due to the symmetry of d we get

(2.6)
$$d(Tx,Ty) = d(Ty,Tx) \leq \lambda d(y,x)^{\alpha} d(y,Ty)^{\beta} d(x,Tx)^{\gamma} \left(\frac{1}{2}\left[d(y,Tx)+d(x,Ty)\right]\right)^{1-\alpha-\beta-\gamma}.$$

Multiplying the inequalities (2.5) and (2.6) it follows that

$$d(Tx,Ty) \leq \lambda d(x,y)^{\alpha} \{d(x,Tx)d(y,Ty)\}^{\frac{\beta+\gamma}{2}} \left(\frac{1}{2}\left[d(y,Tx)+d(x,Ty)\right]\right)^{1-\alpha-\beta-\gamma},$$

which proves that $T \in \Delta_{SH}$ and hence $\Delta_{IH} = \Delta_{SH}$. □

Remark 2.5. From Theorem 2.1, 2.2 and 2.4 it is clear that in each of the Definitions, T can be expressed by fewer constants used as powers in the R.H.S.

Now we consider a version of Ćirić-quasi contraction mapping and show that interpolative contractive mappings are special cases of such type of mappings.

Definition 2.6. Let (X, d) be a metric space. A non-identity mapping $T : X \to X$ is said to be restricted Ćirić-quasi contraction mapping if there exists $\lambda \in [0, 1)$ such that

(2.7) $$d(Tx, Ty) \leq \lambda M(x, y) \text{ for all } x, y \in X \setminus Fix(T),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(y, Tx) + d(x, Ty)]\}$.

Theorem 2.7. *In a complete metric space (X, d), a restricted Ćirić-quasi contraction mapping possesses at least one fixed point in X.*

Proof. The proof is straight forward so we omit the proof. □

Clearly any Ćirić-quasi contraction mapping is also a restricted Ćirić-quasi contraction mapping but the converse is not true in general. The following examples proves our assertion.

Example 2.8. (i) Let $X = [0, 1]$ be the metric space endowed with the usual metric and $T : X \to X$ be defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1-x}{2} & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Then it can be easily checked that T is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, because T has three fixed points 0, $\frac{1}{3}$ and 1.

(ii) Let $X = [1, 2]$ together with the usual metric and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x+1}{2} & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x = 2. \end{cases}$$

Then it can be easily checked that T is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, since T has two fixed points 1 and 2.

(iii) Let $X = [-1, 1]$ be the metric space endowed with the usual metric and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = -1 \\ x & \text{if } -1 < x < 1 \\ -\frac{1}{2} & \text{if } x = 1. \end{cases}$$

Then it can be easily checked that T is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, because T has infinitely many fixed points in X.

Let Δ_{IW} and Δ_{IC} be the collections of all alternate interpolative Berinde weak mappings and restricted Ćirić-quasi contraction mappings respectively. Now we prove the following theorem.

Theorem 2.9. *In a metric space (X, d) if $T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$ then $T \in \Delta_{IC}$.*

Proof. Let $T \in \Delta_{IK}$. Then there exists $\lambda \in [0, 1)$ such that $d(Tx, Ty) \leq \lambda\sqrt{d(x, Tx)d(y, Ty)}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \lambda\sqrt{d(x, Tx)d(y, Ty)} \\ &\leq \lambda\sqrt{M(x, y)^2} = \lambda M(x, y). \end{aligned} \tag{2.8}$$

If $T \in \Delta_{IR}$ then there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^{\frac{1-\alpha}{2}}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^{\frac{1-\alpha}{2}} \\ &\leq \lambda M(x, y)^\alpha \{M(x, y)^2\}^{\frac{1-\alpha}{2}} = \lambda M(x, y). \end{aligned} \tag{2.9}$$

Choose $T \in \Delta_{IH}$. Then there exist $\lambda \in [0, 1)$ and $\alpha, \xi \in (0, 1)$ with $\alpha + 2\xi < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^\xi \left(\frac{1}{2}[d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-2\xi}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we get

$$\begin{aligned} d(Tx, Ty) &\leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^\xi \left(\frac{1}{2}[d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-2\xi} \\ &\leq \lambda M(x, y)^\alpha \{M(x, y)^2\}^\xi (M(x, y))^{1-\alpha-2\xi} = \lambda M(x, y). \end{aligned} \tag{2.10}$$

Consider $T \in \Delta_{IW}$. Then there exists $\lambda \in [0,1)$ such that $d(Tx,Ty) \leq \lambda\sqrt{d(x,y)d(x,Tx)}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we have

$$\begin{aligned} d(Tx,Ty) &\leq \lambda\sqrt{d(x,y)d(x,Tx)} \\ &\leq \lambda\sqrt{M(x,y)^2} = \lambda M(x,y). \end{aligned} \tag{2.11}$$

Hence from (2.8), (2.9), (2.10) and (2.11) we have in any case $T \in \Delta_{IC}$. □

Theorem 2.2 [7], Corollary 1 [8], Theorem 4 [9] and Theorem 1.2 [1] follow from our next corollary.

Corollary 2.10. *In a complete metric space* (X,d) *if* $T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$ *then* T *has a fixed point in* X.

Proof. From Theorem 2.9 we see that if $T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$ then $T \in \Delta_{IC}$. Also Theorem 2.7 says that a mapping $T \in \Delta_{IC}$ always possesses fixed point in X. Hence the corollary. □

Any mapping $T \in \Delta_{IC}$ may not be a member of $\Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$. The next example supports our contention.

Example 2.11. Let us consider $X = [0,1]$ equipped with the usual metric. Also let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x}{2} & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Then clearly T is a restricted Ćirić-quasi contraction mapping for $\frac{1}{2} \leq \lambda < 1$ but not an usual Ćirić-quasi contraction mapping. Also by taking $x = \epsilon$ and $y = 1 - \delta$ with $0 < \epsilon, \delta < 1$ and letting $\epsilon, \delta \to 0$ we see that $T \notin \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$.

In metric fixed point theory, our main objective is to check whether a mapping T over a complete metric space X into itself possesses a fixed point in X. In order to satisfy the interpolative Kannan type contractive condition (1.2) for a mapping $T : X \to X$, we have to know the set $Fix(T)$ and whenever we know the whole set $Fix(T)$ why we bother about, whether the mapping T satisfies the contractive condition (1.2) ?

In one word, to check the existence of fixed points for an interpolative contractive mapping T in X, we have to know the set $Fix(T)$ in advance, which is quite absurd.

Moreover, Theorem 2.7 shows that, if any one of the contractive condition (like Banach, Kannan, Chatterjea) holds "for all $x, y \in X \setminus Fix(T)$" instead of "for all $x, y \in X$" then we can easily remove the part *uniqueness* from the like Theorems (Banach, Kannan, Chaterjea), but in each case we have to know first the set $Fix(T)$.

From this point of view we can conclude that the Theorem 1.3 has no real significance.

To avoid such a situation we can redefine the contractive condition (1.2) in the way that is given below.

Remark 2.12. In a metric space (X,d) if we define an interpolative mapping $T : X \to X$ satisfying

$$d(Tx,Ty) \leq \lambda\sqrt{\max\{d(x,Tx),d(x,y)\}.\max\{d(y,Ty),d(x,y)\}} \tag{2.12}$$

for all $x, y \in X$ and for some $\lambda \in [0,1)$, then it is seen that T can be a non-constant function even if T has a fixed point in X.
Clearly the contractive condition (2.12) can also be taken as

$$d(Tx,Ty) \leq \lambda[\max\{d(x,Tx),d(x,y)\}]^{\alpha}[\max\{d(y,Ty),d(x,y)\}]^{1-\alpha}$$

for all $x, y \in X$, for $\alpha \in (0,1)$ and for some $\lambda \in [0,1)$.

Remark 2.13. It is to be noted that if $Ba(X)$, $Mi(X)$ and $Ci(X)$ are the set of all Banach contractions, interpolative contractive mappings satisfying condition (2.12) and Ćirić quasi contractions on X respectively then $Ba(X) \subset Mi(X) \subset Ci(X)$. Therefore it is clear that in a complete metric space (X,d) an interpolative contractive mapping T satisfying condition (2.12) has a unique fixed point.

3. Interpolative strictly contractive mappings over a compact metric space

In this section we prove some fixed point theorems for interpolative strictly contractive type mappings in the framework of a metric space which is weaker than compact metric space. First we recall the definitions of T-orbitally compact metric space with respect to a self mapping T and orbital continuity of a self mapping over a metric space.

Definition 3.1 ([4])**.** A metric space (X,d) is said to be T-orbitally compact with respect to a mapping $T : X \to X$ if for all $x \in X$, every sequence in the orbit of T at $x \in X$ given by $\mathcal{O}(x,T) = \{x, Tx, T^2x, ...\}$ has a convergent subsequence in X.

Definition 3.2 ([5])**.** Let (X,d) be a metric space. A mapping $T : (X,d) \to (X,d)$ is said to be orbitally continuous if $u \in X$ and such that $u = \lim_{i\to\infty} T^{n_i}x$ for some $x \in X$, then $Tu = \lim_{i\to\infty} TT^{n_i}x$.

Theorem 3.3. *Let (X,d) be a metric space and $T : X \to X$ be a mapping which satisfies*

$$d(Tx,Ty) < \Theta(x,y) \text{ for all } x,y \notin Fix(T), \tag{3.1}$$

where $\Theta(x,y) = \max\{\sqrt{d(x,Tx)d(y,Ty)}, d(x,y)^{\mu}\{d(x,Tx)d(y,Ty)\}^{\frac{1-\mu}{2}},$
$d(x,y)^{\nu}\{d(x,Tx)d(y,Ty)\}^{\tau}\left(\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right)^{1-\nu-2\tau}, d(x,y)^{\xi}\left[\frac{(d(x,Tx)+1)d(y,Ty)}{1+d(x,y)}\right]^{1-\xi}\}$
with $\mu,\nu,\tau,\xi \in (0,1)$ and $\nu+2\tau < 1$. If X is compact (or, T-orbitally compact)

then T has atleast one fixed point in X, provided that T is orbitally continuous in X.

Proof. Let $x_0 \in X$ be chosen as arbitrary. Let us construct an iterative sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \geq 1$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ then x_n will be a fixed point of T. So without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Now from the contractive condition (3.1) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) < \Theta(x_{n-1}, x_n) \text{ for all } n \geq 1. \tag{3.2}$$

Now we have to consider four cases.

Case-I: If $\Theta(x_{n-1}, x_n) = \sqrt{d(x_{n-1}, x_n) d(x_n, x_{n+1})}$ then we get

$$\begin{aligned} & d(x_n, x_{n+1}) < \sqrt{d(x_{n-1}, x_n) d(x_n, x_{n+1})} \\ & \Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \end{aligned} \tag{3.3}$$

Case-II: If $\Theta(x_{n-1}, x_n) = d(x_{n-1}, x_n)^{\mu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\frac{1-\mu}{2}}$ then we have

$$\begin{aligned} & d(x_n, x_{n+1}) < d(x_{n-1}, x_n)^{\mu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\frac{1-\mu}{2}} \\ & \Rightarrow d(x_n, x_{n+1})^{\frac{1+\mu}{2}} < d(x_{n-1}, x_n)^{\frac{1+\mu}{2}} \\ & \Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \end{aligned} \tag{3.4}$$

Case-III: If $\Theta(x_{n-1}, x_n) = d(x_{n-1}, x_n)^{\nu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\tau} \times \left(\frac{1}{2}\left[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\right]\right)^{1-\nu-2\tau}$ then we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &< d(x_{n-1}, x_n)^{\nu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\tau} \left(\frac{1}{2}\left[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\right]\right)^{1-\nu-2\tau} \\ &\leq d(x_{n-1}, x_n)^{\nu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\tau} \left(\frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right)^{1-\nu-2\tau}. \end{aligned} \tag{3.5}$$

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ then from (3.5) it follows that

$$\begin{aligned} d(x_n, x_{n+1}) &< d(x_{n-1}, x_n)^{\nu} \{d(x_{n-1}, x_n) d(x_n, x_{n+1})\}^{\tau} \left(\frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right)^{1-\nu-2\tau} \\ &\leq d(x_n, x_{n+1}), \text{ a contradiction.} \end{aligned} \tag{3.6}$$

Which implies that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$.

Case-IV: If $\Theta(x_{n-1},x_n) = d(x_{n-1},x_n)^{\xi}\left[\frac{(d(x_{n-1},x_n)+1)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right]^{1-\xi}$ then we have

$$\begin{aligned} d(x_n,x_{n+1}) &< d(x_{n-1},x_n)^{\xi}\left[\frac{(d(x_{n-1},x_n)+1)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right]^{1-\xi} \\ &= d(x_{n-1},x_n)^{\xi}d(x_n,x_{n+1})^{1-\xi} \\ \Rightarrow d(x_n,x_{n+1})^{\xi} &< d(x_{n-1},x_n)^{\xi} \end{aligned}$$

$$\Rightarrow d(x_n,x_{n+1}) < d(x_{n-1},x_n). \tag{3.7}$$

Thus from equations (3.3), (3.4), (3.5) and (3.7) we see that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$. So $\{d(x_{n-1},x_n)\}$ is a monotonically decreasing sequence which is bounded below. Therefore there exists some $l \geq 0$ such that $d(x_{n-1},x_n) \to l$ as $n \to \infty$.

Now since X is compact (or, T-orbitally compact), $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, which converges to some $u \in X$. Due to the orbital continuity of T it follows that $\{x_{n_k+1}\}$ converges to Tu and $\{x_{n_k+2}\}$ converges to T^2u respectively. Therefore the continuity of the metric d implies that $\lim_{k\to\infty} d(x_{n_k},x_{n_k+1}) = d(u,Tu)$ and $\lim_{k\to\infty} d(x_{n_k+1},x_{n_k+2}) = d(Tu,T^2u)$. So $d(u,Tu) = l = d(Tu,T^2u)$. If $l > 0$ then $u, Tu \notin Fix(T)$ and therefore

$$d(Tu,T^2u) < \Theta(u,Tu) \text{ implies that } d(Tu,T^2u) < d(u,Tu), \text{ a contradiction.} \tag{3.8}$$

Hence $l = 0$ and $Tu = u$ that is u is a fixed point of T. □

From the above theorem we get the following immediate corollaries.

Corollary 3.4. *Let (X,d) be a metric space and $T : X \to X$ be a mapping which satisfies*

$$d(Tx,Ty) < d(x,Tx)^{\gamma}d(y,Ty)^{1-\gamma} \text{ for all } x,y \notin Fix(T), \gamma \in (0,1). \tag{3.9}$$

If X is compact (or, T-orbitally compact) then T has a fixed point in X, provided that T is orbitally continuous in X.

Example 3.5. Let $X = [0,\infty)$ with the usual metric, $M = \{n + \left(n + \frac{1}{n}\right)^2 : n \geq 2\}$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} n & \text{if } x = n + \left(n+\frac{1}{n}\right)^2, n \geq 2 \\ x & \text{if } x \in X \setminus M. \end{cases}$$

Then T satisfies the contractive condition (3.1) in particular the contractive condition (3.9). Also X is T-orbitally compact and T is orbitally continuous on X. Here we see that T has infinitely many fixed points in X.

Corollary 3.6. *Let (X,d) be a metric space and $T : X \to X$ be a mapping which satisfies*

$$d(Tx,Ty) < d(x,y)^{\gamma}d(x,Tx)^{\delta}d(y,Ty)^{1-\gamma-\delta} \text{ for all } x,y \notin Fix(T), \gamma,\delta \in (0,1). \tag{3.10}$$

If X is compact (or, T-orbitally compact) then T has atleast one fixed point in X, provided that T is orbitally continuous in X.

Corollary 3.7. *Let (X, d) be a metric space and $T : X \to X$ be a mapping which satisfies*

(3.11)

$$d(Tx, Ty) < d(x, y)^{\gamma} d(x, Tx)^{\delta} d(y, Ty)^{\zeta} \left(\frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right]\right)^{1-\gamma-\delta-\zeta}$$

for all $x, y \notin Fix(T)$, where $\gamma, \delta, \zeta \in (0, 1)$ with $\gamma + \delta + \zeta < 1$. If X is compact (or, T-orbitally compact) then T has a fixed point in X, provided that T is orbitally continuous in X.

Corollary 3.8. *Let (X, d) be a metric space and $T : X \to X$ be a mapping which satisfies*

(3.12)

$$d(Tx, Ty) < d(x, y)^{\xi} \left[\frac{(d(x, Tx) + 1)d(y, Ty)}{1 + d(x, y)}\right]^{1-\xi} \text{ for all } x, y \notin Fix(T), \xi \in (0, 1).$$

If X is compact (or, T-orbitally compact) then T has atleast one fixed point in X, provided that T is orbitally continuous in X.

4. A remark on interpolative Kannan contractivity conditions

In [6] the authors have defined (λ, α, β)-interpolative Kannan contraction and prove a fixed point theorem for such mappings. The definition of the mapping is given as follows:

Definition 4.1 ([6])**.** Let (X, d) a metric space and $T : X \to X$ be a self map. T is called a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta} \text{ for all } x, y \in X \setminus Fix(T). \tag{4.1}$$

Theorem 4.2 ([6])**.** *Let (X, d) a complete metric space and $T : X \to X$ be a (λ, α, β)-interpolative Kannan contraction with $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$. Then T has a fixed point in X.*

Theorem 4.2 is not true in general. The next example proves our assertion.

Example 4.3. Let $X = \{\frac{1}{3}, \frac{1}{2}\}$ with usual metric and $T : X \to X$ be given by

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = \frac{1}{3} \\ \frac{1}{3} & \text{if } x = \frac{1}{2}. \end{cases}$$

Then T is a (λ, α, β)-interpolative Kannan contraction with $\lambda = \frac{3}{5}$ and $\alpha = \beta = \frac{1}{3}$. Here X is complete but T has no fixed point in X.

Comment/s: (The reason/s why the proof of Theorem 2 in [6] fails)

In the proof of Theorem 2 (See the line number 5 of Theorem 2 in Page 2 of [6]) the authors used the fact that

(4.2)

$$d(x_n, x_{n+1})^{1-\beta} \leq \lambda d(x_{n-1}, x_n)^{\alpha} \leq \lambda d(x_{n-1}, x_n)^{1-\beta} \text{ whenever } \alpha < 1-\beta,$$

which is actually not true in case $0 < d(x_{n-1}, x_n) < 1$.

Therefore the contractive condition

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha}$$

can not be replaced by the contractive condition (4.1).

E. Karapinar have pointed out a similar idea in Example 2 of [10], where he forewarned about the mappings $T : \{x_0, y_0\} \to \{x_0, y_0\}$ defined by $Tx_0 = y_0$ and $Ty_0 = x_0$. These particular type of mappings defined on two point sets satisfy the contractive condition (3) (See [10]) but are fixed-points free.

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