Applied General Topology

Appl. Gen. Topol. 21, no. 2 (2020), 349-362
doi:10.4995/agt.2020.13943

Discontinuity at fixed point and metric completeness

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Communicated by S. Romaguera

Abstract

In this paper, we prove some new fixed point theorems for a generalized class of Meir-Keeler type mappings, which give some new solutions to the Rhoades open problem regarding the existence of contractive mappings that admit discontinuity at the fixed point. In addition to it, we prove that our theorems characterize completeness of the metric space as well as Cantor's intersection property.

2010 MSC: *47H09; 47H10.*

Keywords: *fixed point; completeness; discontinuity; Cantor's intersection property.*

1. Introduction and Preliminaries

If f is a self-mapping on a complete metric space (X, d) satisfying a contractive condition, then the contractive mapping, in general, ascertains that: For a point $x \in X$, the sequence of iterates is a Cauchy sequence, $\{f^n x\} \to z$, and z is the fixed point of f. However, there exists Meir-Keeler type contractive mapping which ensures the existence of sequence of iterates which is Cauchy and $\{f^n x\} \to z$, but z may not be a fixed point of f. Meanwhile, a more complete study (data dependence, well-posedness, Ulam-Hyers stability, Ostrowski

Received 30 June 2020 – Accepted 30 July 2020

property) was recently proposed in [34].

Fixed point theorems studied by Kannan are considered as the genesis of the question on continuity of contractive mappings at the fixed point (see [15, 16]). Subsequently, the question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the mapping to be continuous at the fixed point was ingeminated as an open problem by Rhoades[33]. In 1999, Pant [27] proved two fixed point theorems in which the considered mappings were discontinuous at the fixed points, hence gave affirmative solutions to the Rhoades problem for both the single and a pair of self-mappings. Some new solutions to this problem with applications to neural networks have been reported in [2, 3, 4, 5, 6, 12, 23, 24, 25, 26, 29, 30, 31, 32, 37, 39]. Fixed point theorems for discontinuous mappings have found a variety of applications, e.g., neural networks are generally used in character recognition, image compression, stock market prediction and to solve non-negative sparse approximation problems ([10, 11, 20, 21, 22, 38]).
Here, we list various classes of Meir-Keeler (M-K) type conditions which ensure convergence of the successive approximations but the limiting point may or may not be a fixed point for the given mappings. For $i \in \{1, ..., 11\}$, we consider:

[M-K] for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that, for any $x, y \in X$,

$$\epsilon \leq m_i(x,y) < \epsilon + \delta \text{ implies } d(fx, fy) < \epsilon.$$

It is obvious that **[M-K]** satisfies the following contractive condition:

$$d(Tx, Ty) < m_i(x, y), \text{ for any } x, y \in X \text{ with } m_i(x, y) > 0, \text{ where}$$

$$m_1(x,y) = d(x,y), \text{ } \textbf{(Meir-Keeler [18])}$$

$$m_2(x,y) = \frac{d(x,fx) + d(y,fy)}{2}, \text{ } \textbf{(Kannan [15])}$$

$$m_3(x,y) = max\{d(x,fx), d(y,fy)\}, \text{ } \textbf{(Bianchini [1])}$$

$$m_4(x,y) = \frac{d(x,fy) + d(y,fx)}{2}, \text{ } \textbf{(Chatterjea [7])}$$

$$m_5(x,y) = \max\Big\{ad(x,fx) + (1-a)d(y,fy), (1-a)d(x,fx) + ad(y,fy)\Big\}, \quad 0 \leq a < 1, \text{ } \textbf{(Pant [28])}$$

$$m_6(x,y) = \max\{d(x,y), d(x,fx), d(y,fy)\}, \text{ } \textbf{(Maiti and Pal [17])}$$

$$m_7(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\Big\}, \text{ } \textbf{(Ćirić [9])}$$

$$m_8(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy) + d(y,fx)}{2}\Big\}, \text{ } \textbf{(Jachymski[14])}$$

$$m_9(x,y)=\max\left\{d(x,y),\frac{k[d(x,fx)+d(y,fy)]}{2},\frac{k[d(x,fy)+d(y,fx)]}{2}\right\},0\le k<1,$$

$$m_{10}(x,y)=\max\Big\{d(x,y),ad(x,fx)+(1-a)d(y,fy),(1-a)d(x,fx)+ad(y,fy),$$
$$\frac{b[d(x,fy)+d(y,fx)]}{2}\Big\},\qquad 0\le a<1 \text{ and } 0\le b<1,$$

(**Bisht and Rakočević** [5])

$$m_{11}(x,y)=\max\Big\{d(x,y),ad(x,fx)+(1-a)d(y,fy),(1-a)d(x,fx)+ad(y,fy),$$
$$\frac{[d(x,fy)+d(y,fx)]}{2}\Big\},0\le a<1.$$

Now, we recall some notions of weaker forms of continuity conditions.

Definition 1.1 ([8, 9])**.** If f is a self-mapping of a metric space (X,d), then the set $O(x,f)=\{f^nx:n=1,2,...\}$ is called the orbit of f at x and f is called orbitally continuous if $u=\lim_i f^{m_i}x$ implies $fu=\lim_i ff^{m_i}x$.

Definition 1.2 ([25])**.** A self-mapping f of a metric space X is called k-continuous, $k=1,2,3,...$, if $f^kx_n\to ft$ whenever, $\{x_n\}$ is a sequence in X such that $f^{k-1}x_n\to t$.

Definition 1.3 ([26])**.** A self-mapping f of a metric space (X,d) is called weakly orbitally continuous if the set $\{y\in X: lim_i f^{m_i}y=u \implies lim_i ff^{m_i}y=fu\}$ is nonempty, whenever the set $\{x\in X: lim_i f^{m_i}x=u\}$ is nonempty.

Remark 1.4. The following observations are now well-established (see [25, 26]):
(i) Continuity implies orbital continuity but not conversely.
(ii) 1-continuity is equivalent to continuity and
$continuity \implies 2-continuity \implies 3-continuity \implies \ldots$, but not conversely.
(iii) Orbital continuity implies weak orbital continuity but the converse need not be true.
(iv) k-continuous mappings are orbitally continuous but the converse need not be true.

The notion of f-orbitally lower semi-continuity was given by Hicks and Rhoades [13].

Definition 1.5. Let (X,d) be a metric space and $f:X\to X$. A mapping $g:X\to\mathbb{R}$ is said to be f-orbitally lower semi-continuous at a point $z\in X$ if $\{x_n\}$ is a sequence in $O(x,f)$ for some $x\in X$, $\lim_{n\to\infty}x_n=z$ implies $\liminf_{n\to\infty}g(x_n)\ge g(z)$.

In [19], the author has shown that the f-orbital lower semi-continuity of $x\to d(x,fx)$ is weaker than orbital continuity and k-continuity of f. The following example illustrates this fact:

Example 1.6. Let $X = \{0,1\} \cup \left\{ \frac{1}{3^n} : n = 1,2,\cdots \right\} \cup \left\{ 1 + \frac{1}{3^n} : n = 1,2,\cdots \right\}$ and d be the usual metric. Define $f : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{4}{3} & \text{if } x = 1 \\ 1 + \frac{1}{3^{n+1}} & \text{if } x = \frac{1}{3^n}, \quad n = 1,2,\cdots \\ \frac{1}{3^n} & \text{if } x = 1 + \frac{1}{3^n}, \quad n = 1,2,\cdots \end{cases}$$

Then

$$O(1,f) = \left\{ 1, 1 + \frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3^2}, \frac{1}{3^2}, \cdots, 1 + \frac{1}{3^n}, \frac{1}{3^n}, \cdots \right\}.$$

Let $\{x_n\}$ be a sequence in $O(1,f)$ with $x_n = \frac{1}{3^n}$ for $n \geq 1$. Then, $x_n \to 0$ as $n \to \infty$. However, $fx_n = 1 + \frac{1}{3^{n+1}} \to 1 \neq f0$. Thus, f is not orbitally continuous. Now, let $\{z_n\}$ be a sequence in X with $z_n = \frac{1}{3^n}$ for $n \geq 1$. Then we have $fz_n = 1 + \frac{1}{3^{n+1}}$ and $f^2 z_n = \frac{1}{3^{n+1}}$. Since $\lim_{n\to\infty} fz_n = 1$ and $\lim_{n\to\infty} f^2 z_n = 0 \neq 4/3 = f1$, the mapping f is not 2-continuous.
Also,

$$g(x) = d(x, fx) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ 1 - \frac{2}{3^{n+1}} & \text{if } x = \frac{1}{3^n}, \quad n = 1,2,\cdots \\ 1 & \text{if } x = 1 + \frac{1}{3^n}, \quad n = 1,2,\cdots \end{cases}$$

Let $x \in X$ and $\{x_n\} \subset O(x,f)$. If $\{x_n\}$ converges, then it converges to 0 or 1. If $\lim_{n\to\infty} x_n = 0$, then

$$\liminf_{n\to\infty} g(x_n) = \begin{cases} 0 \\ \text{or} \\ 1 \end{cases} \geq g(0) = 0$$

. If $\lim_{n\to\infty} x_n = 1$, then $\liminf_{n\to\infty} g(x_n) = 1 > 1/3 = g(1)$. Thus, $x \to d(x,fx)$ is f-orbitally lower semi-continuous.

In this paper, we prove some new fixed point theorems for a generalized class of Meir-Keeler type mappings, which give some new solutions to Rhoades open problem regarding the existence of contractive mappings that admit discontinuity at the fixed point. Further, we prove that our theorems characterize completeness of the metric space as well as Cantor's intersection property.

2. Sequence of successive approximations

Here we show that there exists a large class of contractive mappings which ensure that the sequences of iterates are Cauchy and $\{f^n x\} \to z$, but z may not be a fixed point of f. We begin with the following result:

Proposition 2.1. *Let f be a self-mapping of a complete metric space (X, d) such that for all x, y in X we have*

(i) Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq m_{10}(x,y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$. Then given x in X, the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} f^n x = z$ for some z in X.

Proof. Condition (i) implies that if $m_{10}(x,y) > 0$, then

$$d(fx, fy) < m_{10}(x,y). \tag{2.1}$$

Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_{n+1} = fx_n = f^n x_0$ and $\gamma_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$.

We assume that $x_n \neq x_{n+1}$ for each n. Using (2.1), we get

$$\begin{aligned}
\gamma_n &= d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \\
&< \max\left\{ \begin{array}{c} d(x_{n-1}, x_n), ad(x_{n-1}, fx_{n-1}) + (1-a)\, d(x_n, fx_n), \\ (1-a)d(x_{n-1}, fx_{n-1}) + ad(x_n, fx_n), \dfrac{b[d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})]}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \gamma_{n-1}, a\gamma_{n-1} + (1-a)\,\gamma_n, \\ (1-a)\gamma_{n-1} + a\gamma_n, \dfrac{b[\gamma_{n-1} + \gamma_n]}{2} \end{array} \right\} \\
&= \gamma_{n-1}.
\end{aligned}$$

This shows that $\{\gamma_n\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $\gamma \geq 0$. Suppose $\gamma > 0$. Then there exists a positive integer N such that

$$n \geq N \Rightarrow \gamma < \gamma_n < \gamma + \delta(\gamma). \tag{2.2}$$

By virtue of (i) the above inequality yields $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) < r$. This contradicts with (2.2). Hence, $\gamma_n \to 0$ as $n \to \infty$.

We now prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and a sub-sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$d(x_{n_i}, x_{n_{i+1}}) > 2\varepsilon. \tag{2.3}$$

Select δ in (i) such a way that $0 < \delta \leq \varepsilon$. Since $\lim_{n\to\infty} \gamma_n = 0$, there exists a positive integer N such that

$$\gamma_n < \frac{\delta}{6}, \tag{2.4}$$

whenever $n \geq N$. Let $n_i > N$. Then there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that

$$d(x_{n_i}, x_{m_i}) \geq \varepsilon + \frac{\delta}{3}. \tag{2.5}$$

If not, then using (2.4) and (2.5), we have

$$\begin{aligned} d(x_{n_i}, x_{n_{i+1}}) &\leq d(x_{n_i}, x_{n_{i+1}-1}) + d(x_{n_{i+1}-1}, x_{n_{i+1}}) \\ &< \varepsilon + \frac{\delta}{3} + \frac{\delta}{6} = \varepsilon + \frac{\delta}{2} < 2\varepsilon, \end{aligned}$$

a contradiction with (2.3). Let m_i^* be the smallest integer such that $n_i < m_i^* < n_{i+1}$ and

$$d(x_{n_i}, x_{m_i^*}) \geq \varepsilon + \frac{\delta}{3}. \tag{2.6}$$

Then $d(x_{n_i}, x_{m_i^*-1}) < \varepsilon + \frac{\delta}{3}$. In view of (2.1) and (2.2), we get

$$\begin{aligned} \varepsilon &< \varepsilon + \frac{\delta}{3} \leq d(x_{n_i}, x_{m_i^*}) = d(fx_{n_i-1}, fx_{m_i^*-1}) < m_{10}(x_{n_i-1}, x_{m_i^*-1}) \\ &= \max \left\{ \begin{array}{c} d\left(x_{n_i-1}, x_{m_i^*-1}\right), ad(x_{n_i-1}, fx_{n_{i-1}}) + (1-a)d(x_{m_i^*-1}, fx_{m_i^*-1}), \\ (1-a)d(x_{n_i-1}, fx_{n_{i-1}}) + ad(x_{m_i^*-1}, fx_{m_i^*-1}), \\ \dfrac{b[d(x_{n_i-1}, fx_{m_i^*-1}) + d(x_{m_i^*-1}, fx_{n_i-1})]}{2} \end{array} \right\} \\ &\leq \max\left\{d(x_{n_i-1}, x_{m_i^*-1}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i^*-1}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})\right\} \\ &\leq d(x_{n_i}, x_{m_i^*-1}) + d(x_{n_i-1}, x_{n_i}) \\ &< \varepsilon + \frac{\delta}{3} + \frac{\delta}{6} = \varepsilon + \frac{\delta}{2}, \end{aligned}$$

that is,

$$\varepsilon + \frac{\delta}{3} \leq m(x_{n_i-1}, x_{m_i^*-1}) < \varepsilon + \frac{\delta}{2}.$$

By virtue of (i), the last inequality yields $d(fx_{n_i-1}, fx_{m_i^*-1}) < \varepsilon$, i.e., $d(x_{n_i}, x_{m_i^*}) < \varepsilon$. This contradicts (2.6) and, hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x = z$. □

We now present two examples which satisfy the above proposition but f is fixed point free.

Example 2.2. Let $X = [1,2] \cup \left\{1 - \dfrac{1}{3^n} : n = 0, 1, 2, \cdots\right\}$ and d be the usual metric. Define $f : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } 1 \leq x \leq 2. \\ 1 - \dfrac{1}{3^{n+1}} & \text{if } x = 1 - \dfrac{1}{3^n}, \qquad n = 0, 1, 2, \cdots. \end{cases}$$

Then

$$f(X) = \left\{1 - \frac{1}{3^n} : n = 0, 1, 2, \cdots\right\}$$

and f is fixed point free. The mapping f satisfies the contractive condition (i) with

$$\delta(\varepsilon) = \begin{cases} \frac{1}{3^n} - \varepsilon & if \quad \frac{1}{3^{n+1}} \leq \varepsilon < \frac{1}{3^n}, n = 0, 1, 2, \cdots \\ \varepsilon & if \qquad \varepsilon \geq 1. \end{cases}$$

Example 2.3. Let $X = [0, 2]$ equipped with the Euclidean metric d. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{1+3x}{4} & if \quad 0 \le x < 1 \\ 0 & if \quad x \ge 1 \end{cases}.$$

Then X is a complete metric space and f satisfies the contractive condition (i) with

$$\delta(\varepsilon) = \begin{cases} \frac{\varepsilon}{3} & if \quad 0 \le \varepsilon \le \frac{3}{4} \\ 1 - \varepsilon & if \quad \frac{3}{4} < \varepsilon < 1 \\ \varepsilon & if \quad \varepsilon \ge 1 \end{cases}$$

but does not posses a fixed point. It is easy to verify that for each x in X, the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $f^n x \to 1$ (see [29]).

Replacing $m_{10}(x, y)$ given in the condition (i) of Proposition 2.1 by $m_{11}(x, y)$ and using similar arguments, proof of the following proposition follows easily.

Proposition 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping such that for all $x, y \in X$ we have*
$(i)'$ Given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon \le m_{11}(x, y) < \varepsilon + \delta \Longrightarrow d(fx, fy) < \varepsilon.$$

Then the sequence of iterates $\{f^n x\}$ is a Cauchy sequence for a given $x \in X$ and $\lim_{n\to\infty} f^n x = z$ for some $z \in X$.

3. Discontinuity at fixed point

We give a new solution to the problem of continuity at the fixed point for the number $m_{10}(x, y)$.

Theorem 3.1. *Let f be a self-mapping of a complete metric space (X, d) such that*

$$(i) \text{ Given } \varepsilon > 0 \text{ there exists a } \delta = \delta(\varepsilon) > 0 \text{ such that}$$
$$\varepsilon \le m_{10}(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

for all x, y in X. If $x \to d(x, fx)$ is f-orbitally lower semi-continuous, then f has a unique fixed point $z \in X$ and $f^n x \to z$ as $n \to \infty$. Moreover, f is continuous at z if and only if $\lim_{x\to z} \max \{d(x, fx), d(z, fz)\} = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$.

Proof. Let x be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f^n x, n = 0, 1, 2, 3....$ Then following the proof of Proposition 2.1 above, we get that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$.

Since $x_n \to z$ satisfying $d(x_n, fx_n) = d(f^n x, f^{n+1} x) \to 0$ as $n \to \infty$, by f-orbital lower semi-continuity of $x \to d(x, fx)$, one gets

$$d(z, fz) \le \liminf_{n\to\infty} d(x_n, fx_n) = 0,$$

which implies that $z = fz$, i.e., z is a fixed point of f. Uniqueness of the fixed point follows easily.

Let f be continuous at the fixed point z. For $\lim_{x\to z} \max\{d(x,fx), d(z,fz)\} = 0$; let $y_n \to z$. Then $fy_n \to fz = z$, $d(y_n, fy_n) = 0$ and

$$\lim_{n\to\infty} \max\{d(y_n, fy_n), d(z,fz)\} = 0.$$

Conversely, let $\lim_{x\to z} \max\{d(x,fx), d(z,fz)\} = 0$. To show that f is continuous at the fixed point z, let $y_n \to z$. Then, we have

$$\lim_{n\to\infty} \max\{d(y_n, fy_n), d(z,fz)\} = 0.$$

This implies that $\lim_{n\to\infty} d(y_n, fy_n) = 0$ and, hence, $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} y_n = z = fz$. Therefore, f is continuous at the fixed point z. This completes the proof. □

Theorem 3.2. *Let f be a self-mapping of a complete metric space (X,d) such that*

(i) *Given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*
$$\varepsilon \le m_{10}(x,y) < \varepsilon + \delta \Rightarrow d(fx,fy) < \varepsilon,$$

for all x, y in X. Then f possesses a fixed point if and only if f is weakly orbitally continuous. Moreover, the fixed point is unique and f is continuous at z if and only if $\lim_{x\to z} \max\{d(x,fx), d(z,fz)\} = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz,fx) = 0$.

Proof. Let x be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f^n x, n = 0,1,2,3....$ Then following the proof of Theorem 3.1 above, we get that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$.

Suppose that f is weakly orbitally continuous. Since $f^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f we get, $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some $y_0 \in X$. This implies that $z = fz$ since $f^{n+1} y_0 \to z$. Therefore z is a fixed point of f.

Conversely, suppose that the mapping f has a fixed point, say z. Then $\{f^n z = z\}$ is a constant sequence such that $\lim_n f^n z = z$ and $\lim_n f^{n+1} z = z = fz$. Hence, f is weak orbitally continuous. Rest of the the proof follows from the proof of Theorem 3.1. □

Remark 3.3. The last part of Theorem 3.1 or Theorem 3.2 can alternatively be stated as: f is discontinuous at z if and only if $\lim_{x\to z} \max\{d(x,fx), d(z,fz)\} \ne 0$ or equivalently, $\lim_{x\to z} \sup d(fz,fx) > 0$.

Remark 3.4. Theorems 3.1 and 3.2 hold true if we replace $m_{10}(x,y)$ given in (i) by $m_{11}(x,y)$.

The following theorem is a consequence of above theorems.

Theorem 3.5. *Let f be a self-mapping of a complete metric space (X,d) such that*

(i) Given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that
$$\varepsilon \leq m_3(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

for all x, y in X. Then f possesses a fixed point if and only if f is weakly orbitally continuous or if $x \to d(x, fx)$ is f-orbitally lower semi-continuous, then f has a fixed point $z \in X$ and $f^n x \to z$ as $n \to \infty$. Moreover, the fixed point is unique and f is continuous at z if and only if $\lim\limits_{x\to z} \sup d(fz, fx) = 0$.

The following result is an easy consequence of Theorem 3.1.

Corollary 3.6. *Let f be a self-mapping of a complete metric space (X, d) satisfying (i) of Theorem 3.1 for all x, y in X. If f is k-continuous for some $k \geq 1$ or if f is orbitally continuous then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim\limits_{x\to z} \sup d(fz, fx) = 0$.*

W now give an example to illustrate the above result.

Example 3.7. Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{1+x}{2} & if \quad 0 \leq x \leq 1 \\ \frac{x-1}{2} & if \quad 1 < x \leq 2 \end{cases}.$$

Then f satisfies all the conditions of Theorem 3.5 and has a unique fixed point $z = 1$ at which f is discontinuous [29]. One can compute that

$$g(x) = d(x, fx) = \begin{cases} \frac{x-1}{2} & if \quad 0 \leq x \leq 1 \\ \frac{1+x}{2} & if \quad 1 < x \leq 2 \end{cases}.$$

The mapping $g(x) = x \to d(x, fx)$ is f-orbitally lower semi-continuous and satisfies the condition (i) with

$$\delta(\varepsilon) = \begin{cases} \varepsilon & if \quad \varepsilon \leq \frac{1}{2} \\ 1-\varepsilon & if \quad \frac{1}{2} < \varepsilon < 1 \\ \varepsilon & if \quad \varepsilon \geq 1 \end{cases}.$$

Here, $\lim\limits_{x\to z} \max\{d(x, fx), d(z, fz)\}$ does not exist. Also, $\lim\limits_{x\to z} \sup d(fz, fx) = 1$.

Theorem 3.8. *Let f be a self-mapping of a complete metric space (X, d) such that*

(ii) Given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that
$$\varepsilon \leq m_{10}(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

for all x, y in X. If $a = b = 0$, then f has a unique fixed point whenever $x \to d(x, fx)$ is f-orbitally lower semi-continuous or f is weakly orbitally continuous. If $a, b > 0$, then f possesses a unique fixed point at which f is continuous.

Proof. The proof follows on the same lines of the proof of Theorem 3.1 above. As seen in Theorem 3.1 above, f need not be continuous at the fixed point if $a = 0 = b$. We now show that f is continuous at the fixed point when $a, b > 0$. Suppose $a > 0$ and z is the fixed point of f. Let $\{x_n\}$ be any sequence in X

such that $x_n \to z$ as $n \to \infty$. Then using (ii), for sufficiently large values of n, we get

$$d(z, fx_n) = d(fz, fx_n)$$

$$< \max\Big\{d(z, x_n), ad(z, fz) + (1-a)\, d(x_n, fx_n),$$

$$(1-a)d(z, fz) + ad(x_n, fx_n), \frac{b[d(z, fx_n) + d(x_n, fz)]}{2}\Big\}$$

$$= \max\Big\{d(z, x_n), (1-a)\, d(x_n, fx_n), ad(x_n, fx_n), \frac{b[d(z, fx_n) + d(x_n, fz)]}{2}\Big\}$$

$$\leq \max\left\{\epsilon_1, \epsilon_2 + (1-a)\, d(z, fx_n), \epsilon_3 + ad(z, fx_n), \frac{b[d(z, fx_n) + \epsilon_4]}{2}\right\},$$

where $\epsilon_i (i = 1, 2, 3, 4) \to 0$ as $n \to \infty$. This yields $d(z, fx_n) < \epsilon_1$, $ad(z, fx_n) < \epsilon_2$, $(1-a)d(z, fx_n) < \epsilon_3$ or $\dfrac{bd(z, fx_n)}{2} < \epsilon_4$. On letting $n \to \infty$, we get $\lim_{n\to\infty} fx_n = z = fz$. Hence f is continuous at the fixed point. □

4. Characterization of metric completeness

In a complete metric space, the following well-known variant of Cantor's intersection theorem holds.

Theorem 4.1. *Suppose that X is a complete metric space, and C_n is a sequence of non-empty closed nested subsets of X whose diameters tend to zero:*

$\lim_{n\to\infty} diam(C_n) = 0$, *where* $diam(C_n)$ *is defined by* $diam(C_n) = \sup\{d(x, y)| x, y \in C_n\}$. *Then the intersection of the C_n contains exactly one point:*

$\bigcap_{n=1}^{\infty} C_n = \{x\}$ *for some* $x \in X$.

A converse to this theorem is also true: if X is a metric space with the property that the intersection of any nested family of non-empty closed subsets whose diameters tend to zero is non-empty, then X is a complete metric space.

In the next result, we show that Theorem 3.5 characterizes metric completeness of X. However, there is a substantive difference between the next theorem and similar theorems (e. g., Subrahmanyam [35], Suzuki [36]) giving characterization of completeness in terms of fixed point property for contractive type mappings. Subrahmanyam [35] and Suzuki [36] have shown that the contractive condition implies continuity at the fixed point; and completeness of the metric space X is equivalent to the existence of fixed point. In the next theorem motivated by ([25, 26]) we prove that completeness of the space is equivalent to fixed point property for a larger class of mappings including continuous as well as discontinuous mappings.

Theorem 4.2. *If every $x \to g(x) = d(x, fx)$ is f-orbitally lower semi-continuous or weak orbitally continuous self-mapping f of a metric space (X, d) satisfying the condition (i) of Theorem 3.5 has a fixed point, then X is complete.*

Proof. Suppose that every $x \to d(x, fx)$ is f-orbitally lower semi-continuous or weak orbitally continuous self-mapping f of a metric space (X, d) satisfying the condition (i) of Theorem 3.5 possesses a fixed point. We will prove that X is complete. Arguing by contradiction, suppose that X is not complete. Then there exists a Cauchy sequence in X, say $M = \{u_n\}_{n\in\mathbb{N}}$ having distinct points which does not converge in X. Let $x \in X$ be any arbitrary point. Then, since x is not a limit point of the Cauchy sequence M, and we have $d(x, M - \{x\}) > 0$ and there exists an integer $n_x \in \mathbb{N}$ such that $x \neq u_{n_x}$ and for each $m \geq n_x$

$$d(u_{n_x}, u_m) < \frac{1}{2}d(x, u_{n_x}). \tag{4.1}$$

Consider a mapping $f : X \mapsto X$ by $f(x) = u_{n_x}$. Then, $f(x) \neq x$ for each x and, using (4.1), for any $x, y \in X$ we get

$$d(fx, fy) = d(u_{n_x}, u_{n_x=y}) < \frac{1}{2}d(x, u_{n_x}) = d(x, fx), \text{if } \ n_x \leq n_y,$$

or

$$d(fx, fy) = d(u_{n_x}, u_{n_y}) < \frac{1}{2}d(y, u_{n_y}) = d(y, fy), \text{if } \ n_x > n_y.$$

This implies that

$$d(fx, fy) < \frac{1}{2}max\{d(x, fx), d(y, fy)\}. \tag{4.2}$$

Or equivalently, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon \leq max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon. \tag{4.3}$$

It is clear from (4.2) and (4.3) that the mapping f satisfies condition (i) of Theorem 3.5. Moreover, f is a fixed point free mapping whose range is contained in the non-convergent Cauchy sequence M. Hence, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{fx_n\}_{n\in\mathbb{N}}$ converges, i.e., there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which the condition $f^{k-1}x_n \to t \Rightarrow f^k x_n \to ft$ for $k > 1$ is violated. Therefore, f is k-continuous mapping. Since $x_n \in O(x, f)$, it is of the form $x_n = f^{i_n}x$ with $i_n \in \mathbb{N}$. Set $y_n = f^{i_n+1-k}x$ when $i_n \geq k-1$. Then, $f^{k-1}y_n = x_n \to t$ as $n \to \infty$. By k-continuity of f, we get $fx_n = f^k y_n \to ft$ as $n \to \infty$. Thus, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} d(x_n, fx_n) = d(t, ft) = g(t)$, which implies that g is f-orbitally lower semi-continuous at t. In a similar manner it follows that f is weak orbitally continuous. Thus, we have a self-mapping f of X which satisfies all the conditions of Theorems 3.5 but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence X is complete. □

We now show that Theorem 3.5 characterizes Cantor's intersection property.

Theorem 4.3. *Let (X, d) be a metric space and f a self-mapping of X satisfying condition (i) of Theorem 3.5. Suppose X satisfies Cantor's intersection property and $x \to d(x, fx)$ is f-orbitally lower semi-continuous or f is weakly orbitally continuous. Then f has a fixed point and f is continuous at z if and only if* $\lim_{x\to z} max\{d(x, fx), d(z, fz)\} = 0$.

Proof. Let x be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f^n x, n = 0, 1, 2, 3....$ Then following the proof of Proposition 2.1 above, we get that $\{x_n\}$ is a Cauchy sequence. Define a sequence $\{T_n : n = 1, 2, 3, ...\}$ of nonempty subsets of X by $T_n = \{x_i : i \geq n\}$. Let C_n denote the closure of $T_n \in X$. Then for each n it is obvious that C_n is a nonempty closed subset of X, $C_{n+1} \subseteq C_n$ and $\lim_{n\to\infty} diam(C_n) = 0$. Since Cantor's intersection property holds in X, $\cap\{C_n\}$ consists of exactly one point, say z, which is nothing but the limit of the Cauchy sequence $\{x_n\}$. Rest of the proof follows from Theorem 3.5. □

Theorem 4.4. *Let (X, d) be a metric space. If every $x \to d(x, fx)$ is f-orbitally lower semi-continuous or weak orbitally continuous self-mapping f of a metric space (X, d) satisfying the condition (i) of Theorem 3.5 has a fixed point, then X satisfies Cantor's intersection property.*

Proof. Suppose that every $x \to d(x, fx)$ is f-orbitally lower semi-continuous or weak orbitally continuous self-mapping f of a metric space (X, d) satisfying the condition (i) of Theorem 3.5 possesses a fixed point. We assert that X satisfies Cantor's intersection property. If possible, suppose X does not satisfy Cantor's intersection property, then there exists a sequence $\{C_n\}$of nonempty closed subsets of X satisfying $C_{n+1} \subseteq C_n$ and $\lim_{n\to\infty} diam(C_n) = 0$ and having empty intersection. Construct a sequence $T = \{x_n\} \in X$ such that $x_i \in A_i$. $\lim_{n\to\infty} diam(C_n) = 0$, given $\epsilon > 0$ there exists a positive integer N such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Therefore $\{x_n\}$ is a Cauchy sequence. However, $T = \{x_n\}$ is a non-convergent Cauchy sequence since the sequence $\{C_n\}$ has empty intersection. As done in the proof of Theorem 4.2 we can now define f-orbitally lower semi-continuous or weakly orbitally continuous self-mapping on a metric space (X, d) satisfying condition (i) which does not possess a fixed point. This contradicts our hypothesis. Therefore, Cantor's intersection property holds in X. □

Combining Theorem 4.2 and Theorem 4.4, we get the following theorem:

Theorem 4.5. *For a metric space (X, d), the following are equivalent:*
(a) X is complete.
(b) X satisfies Cantor's intersection property.
(c) every $x \to d(x, fx)$ is f-orbitally lower semi-continuous or weak orbitally continuous self-mapping f of a metric space (X, d) satisfying the condition:

$$\text{Given } \varepsilon > 0 \text{ there exists a } \delta = \delta(\varepsilon) > 0 \text{ such that}$$
$$\varepsilon \leq m_3(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

for all x, y in X, has a fixed point.

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