Further remarks on group-2-groupoids

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ABSTRACT

The aim of this paper is to obtain a group-2-groupoid as a 2-groupoid object in the category of groups and also as a special kind of an internal category in the category of group-groupoids. Corresponding group-2-groupoids, we obtain some categorical structures related to crossed modules and group-groupoids and prove categorical equivalences between them. These results enable us to obtain 2-dimensional notions of group-groupoids.

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1. INTRODUCTION

There are several 2-dimensional notions of groupoids such as double groupoids, 2-groupoids, and crossed modules over groupoids. The purpose of this paper is to obtain 2-dimensional notions of group-groupoids which are internal groupoids in the category of groups and widely used under the name of 2-groups.

The term "categorification", which was first used by Louis Crane [13] in the context of mathematical physics, is the process of replacing set-theoretic theorems by category-theoretic concepts. The aim of categorification is to develop a richer case of existing mathematics by replacing sets with categories, functions with functors and equations between functions with natural isomorphisms between functors. In this approach, the categorified version of a group is called a group-groupoid [2, 5]. Group-groupoids, which are also known as $\mathcal{G}$-groupoids [6] or 2-groups [4], are internal categories (hence internal groupoids) in the
category $\text{Gp}$ of groups \cite{22, 23}. Equivalently, group-groupoids can be thought as group objects in the category $\text{Cat}$ of small categories \cite{6, 23}.

Another useful viewpoint of group-groupoids is to think them as crossed modules over groups. Crossed modules which can be viewed as 2-dimensional groups \cite{7} are widely used in homotopy theory \cite{8}, homological algebra \cite{16}, and algebraic K-theory \cite{21}. The well-known categorical equivalence between crossed modules and group-groupoids is proved by Brown and Spencer \cite{6}. This equivalence is introduced in \cite{4} by obtaining a group-groupoid as a 2-category with a unique object. Crossed modules, and their higher dimensional analogues, provide algebraic models for homotopy n-types; the group-2-groupoids of this paper in principle provide algebraic models for certain homotopy 3-types.

In the previous paper \cite{1}, the notions of a group-2-groupoid were introduced and compared with a corresponding structure related to crossed modules over groups. On the other hand, the main objective of this paper is to obtain the structure of a group-2-groupoid as a 2-groupoid object in the category of groups and also as a special kind of internal category in the category of group-groupoids. In section 4, we present the notion of crossed modules over group-groupoids and prove that there is a categorical equivalence between group-2-groupoids and crossed modules over group-groupoids using the categorical equivalence between 2-groupoids and crossed modules over groupoids given in \cite{17}. In section 5, we show that group-2-groupoids are categorically equivalent to special kind of internal categories in the category of crossed modules.

2. Preliminaries

Let $\mathcal{C}$ be a finitely complete category and $D_0, D_1$ are objects of the ambient category $\mathcal{C}$. An internal category $\mathcal{D} = (D_0, D_1, s, t, \varepsilon, m)$ in $\mathcal{C}$ consists of an object $D_0$ in $\mathcal{C}$ called the object of objects and an object $D_1$ in $\mathcal{C}$ called the object of arrows (i.e. morphisms), together with morphisms $s, t: D_1 \to D_0$, $\varepsilon: D_0 \to D_1$ in $\mathcal{C}$ called the source, the target and the identity maps, respectively,

$$D_1 \xrightarrow{s} D_0 \xleftarrow{t}$$

such that $s \varepsilon = t \varepsilon = 1_{D_0}$ and a morphism $m: D_1 \times_{D_0} D_1 \to D_1$ of $\mathcal{C}$ called the composition map (usually expressed as $m(f, g) = g \circ f$) where $D_1 \times_{D_0} D_1$ is the pullback of $s, t$ such that $\varepsilon s(f) \circ f = f = f \circ \varepsilon s(f)$ \cite{22}. An internal groupoid in $\mathcal{C}$ is an internal category with a morphism $\eta: D_1 \to D_1$, $\eta(f) = \overline{f}$ in $\mathcal{C}$ called inverse such that $\overline{f} \circ f = 1_{s(f)}, \ f \circ \overline{f} = 1_{t(f)}$.

We write $C(x, y)$ for all morphisms from $x$ to $y$ where $x, y \in C_0$. If $C(x, y) = \emptyset$ for all $x, y \in C_0$ such that $x \neq y$, then $\mathcal{C}$ is called totally disconnected category.

We introduce the definition of a 2-category as given in \cite{4}. A 2-category $\mathcal{C} = (C_0, C_1, C_2)$ consists of a set of objects $C_0$, a set of 1-morphisms $C_1$, and
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A set of 2-morphisms $C_2$ as follows:

\[
\begin{array}{ccc}
  & f & \\
  x & \downarrow \alpha & y \\
  & g & \\
\end{array}
\]

with maps $s: C_1 \to C_0$, $s(f) = x$, $s_h: C_2 \to C_0$, $s_h(\alpha) = x$, $s_v: C_2 \to C_1$, $s_v(\alpha) = f$, $t: C_1 \to C_0$, $t(f) = y$, $t_h: C_2 \to C_0$, $t_h(\alpha) = y$, $t_v: C_2 \to C_1$, $t_v(\alpha) = g$, called the source and the target maps, respectively, the composition of 1-morphisms as in an ordinary category, the associative horizontal composition of 2-morphisms $\circ_h: C_2 \times C_0 C_2 \to C_2$ as

\[
\begin{array}{ccc}
  & f & \downarrow \alpha & \beta & y \\
  x & \downarrow & & & \\
  & g & & & \\
\end{array}
\]

where $C_2 \times C_0 C_2 = \{(\alpha, \delta) \in C_2 \times C_2 | s_h(\delta) = t_h(\alpha)\}$ and the associative vertical composition of 2-morphisms $\circ_v: C_2 \times C_1 C_2 \to C_2$ as

\[
\begin{array}{ccc}
  & f & \downarrow \alpha & \beta & y \\
  x & \downarrow & & & \\
  & h & & & \\
\end{array}
\]

where $C_2 \times C_1 C_2 = \{(\alpha, \beta) \in C_2 \times C_2 | s_v(\beta) = t_v(\alpha)\}$ such that satisfying the following interchange rule:

\[
(\theta \circ_v \delta) \circ_h (\beta \circ_v \alpha) = (\theta \circ_h \beta) \circ_v (\delta \circ_h \alpha)
\]

whenever one side makes sense, and the identity maps $\varepsilon: C_0 \to C_1, \varepsilon(x) = 1_x$, $\varepsilon_h: C_0 \to C_2$, $\varepsilon_h(x) = 1_{1_x}$, such that $\alpha \circ_h 1_{1_x} = \alpha = 1_{1_g} \circ_h \alpha$ and $\varepsilon_v: C_1 \to C_2$, $\varepsilon_v(f) = 1_f$ such that $\alpha \circ_v 1_f = \alpha = 1_g \circ_v \alpha$. Therefore, the construction of a 2-category $\mathcal{C} = (C_0, C_1, C_2)$ contains compatible category structures $C_1 = (C_0, C_1, s, t, \varepsilon, \circ_i)$, $C_2 = (C_0, C_2, s_h, t_h, \varepsilon_h, \circ_i)$, and $C_3 = (C_1, C_2, s_v, t_v, \varepsilon_v, \circ_v)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
  & C_2 & \to C_1 \\
  & \downarrow \alpha & \downarrow \beta \\
  & C_0 & \to C_0 \downarrow \varepsilon_h \\
\end{array}
\]

Let $\mathcal{C}$ and $\mathcal{C}'$ be 2-categories. A 2-functor is a map $F: \mathcal{C} \to \mathcal{C}'$ sending each object of $\mathcal{C}$ to an object of $\mathcal{C}'$, each 1-morphism of $\mathcal{C}$ to 1-morphism of $\mathcal{C}'$ and...
2-morphism of $C$ to 2-morphism of $C'$ as follows:
\[ x \xrightarrow{f \downarrow \alpha} y \mapsto F(x) \xrightarrow{F(f) \downarrow F(\alpha)} F(y) \]
such that $F(f_1 \circ f) = F(f_1) \circ F(f)$, $F(\delta \circ_h \alpha) = F(\delta) \circ_h F(\alpha)$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)$, $F(1_{1_x}) = 1_{F(1_x)} = 1_{F(1)}$, $F(1_f) = 1_{F(f)}$. Hence 2-categories form a category which is denoted by $\textbf{2Cat}$ [24].

A strict 2-groupoid is a 2-category all of whose 1-morphisms are invertible and in which all 2-morphisms are invertible horizontally and vertically.

\[ x \xrightarrow{f \downarrow \alpha} y \xmapsto{\tilde{f} \downarrow \alpha^\top} x, \xrightarrow{1_x \downarrow 1_x} x, \xrightarrow{f \downarrow \alpha} y = x \xrightarrow{f \downarrow \alpha} y \]

Let $\mathcal{G}, \mathcal{G}'$ be 2-groupoids. A morphism of 2-groupoids is a 2-functor $F: \mathcal{G} \to \mathcal{G}'$ which preserves the 2-groupoid structures. Thus, 2-groupoids and their morphisms form a category which is denoted by $\textbf{2Gpd}$ [24].

A group-groupoid is an internal category in $\textbf{Gp}$ [22]. Also, a group-groupoid can be obtained as a group object in the category $\textbf{Cat}$ of small categories (or in $\textbf{Gpd}$). A morphism of group-groupoids is a morphism of groupoids which preserves group structures. Hence we can define the category of group-groupoids, which is denoted by $\textbf{2Gp}$ or $\textbf{GpGd}$. For further details about group-groupoids, see [24, 6, 4].

By a crossed module as defined by Whitehead, it is meant a pair $M, N$ of groups together with an action $\bullet: N \times M \to M$ of groups and a morphism $\partial: M \to N$ of groups such that $\partial(n \cdot m) = n\partial(m)n^{-1}$ and $\partial(m) \cdot m' = mm'\partial(m)^{-1}$ [28, 29].

Let $K = (M, N, \partial, \bullet)$, $K' = (M', N', \partial', \bullet')$ be crossed modules and $\lambda_1: N \to N'$, $\lambda_2: M \to M'$ be morphisms of groups. If $\lambda_1, \lambda_2$ satisfies the conditions $\lambda_1\partial = \partial'\lambda_2$ and $\lambda_2(n \cdot m) = \lambda_1(n) \bullet \lambda_2(m)$, then $(\lambda_2, \lambda_1): K \to K'$ is called morphism of crossed modules [6]. Hence crossed modules and their morphisms form a category which we denote by $\textbf{Cm}$.

The following theorem was proved by Brown and Spencer in [6]:

**Theorem 2.1.** The category of group-groupoids and the category of crossed modules are equivalent.

Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids over the same object set $X$ such that $\mathcal{H}$ is totally disconnected. We recall from [8, 17, 11] that an action
of $G$ on $H$ is a partially defined map
\[ \bullet : G \times H \to H, \quad (g, h) \mapsto g \bullet h \]
such that the following conditions satisfies

\begin{itemize}
  \item [AG 1] $g \bullet h$ is defined iff $t(h) = s(g)$, and $t(g \bullet h) = t(g)$,
  \item [AG 2] $(g_2 \circ g_1) \bullet h = g_2 \bullet (g_1 \bullet h)$,
  \item [AG 3] $g \bullet (h_2 \circ h_1) = (g \bullet h_2) \circ (g \bullet h_1)$, for $h_1, h_2 \in H(x, x)$ and $g \in G(x, y)$,
  \item [AG 4] $1_x \bullet h = h$, for $h \in H(x, x)$.
\end{itemize}

From these conditions, it can be easily obtain that $g \bullet 1_x = 1_y$, for $g \in G(x, y)$.

Using this action of $G$ on $H$, we can obtain a groupoid which is called semi-direct product of $G$ and $H$ denoted by $G \rtimes H$. Let $x \xrightarrow{g} y \xrightarrow{h} y$ are morphisms of $G$ and $H$, respectively, then $(g, h)$ is a morphism as follows
\[ x \xrightarrow{(g,h)} y \]
where the structure maps are defined by $s(g, h) = s(g)$, $t(g, h) = t(g)$, $\varepsilon(x) = (1_x, 1_x)$. If
\[ x \xrightarrow{g} y \xrightarrow{h} y \xrightarrow{g_1} z \xrightarrow{h_1} z \]
then the composition of morphisms is defined by
\[ (g_1, h_1) \circ (g, h) = (g_1 \circ g, h_1 \circ (g_1 \bullet h)) \]

The notion of crossed modules over groupoids is introduced by Brown-Higgins [9, 10] and Brown-Icen [11]. Let $G = (X, G)$ and $H = (X, H)$ be groupoids over the same object set $X$ such that $H$ is totally disconnected. A crossed module $K = (H, G, \partial, \bullet)$ over groupoids consists of a morphism $\partial = (1, \partial) : H \to G$ of groupoids which is identity on objects together with an action $\bullet : G \times H \to H$ of groupoids which satisfies $\partial(g \bullet h) = g \circ \partial(h) \circ \overline{g}$ and $\partial(h) \bullet h_1 = h \circ h_1 \circ \overline{h}$, for $h, h_1 \in H(x, x)$ and $g \in G(x, y)$.

Let $K = (H, G, \partial, \bullet)$ and $K' = (H', G', \partial', \bullet')$ be crossed modules over groupoids. A morphism of crossed modules over groupoids is a mapping $\lambda = (\lambda_2, \lambda_1, \lambda_0) : K \to K'$ which satisfies $\lambda_2 \partial = \partial' \lambda_1$ and $\lambda_1(g \bullet h) = \lambda_2(g) \bullet \lambda_1(h)$ where $(\lambda_0, \lambda_1) : H \to H'$ and $(\lambda_0, \lambda_2) : G \to G'$ are morphisms of groupoids. Hence the category of crossed modules over groupoids can be defined which we denoted by $\text{Cmg}$.

The following result was proved by Icen in [17]. Since we need some details in section 4, we give a sketch proof in terms of our notations.

**Theorem 2.2.** The categories of 2-groupoids and of crossed module over groupoids are equivalent.

**Proof.** For any 2-groupoid $G = (G_0, G_1, G_2)$, we know that $B = (G_0, G_1)$ is a groupoid. Let $A(x) = \{ \alpha \in G_2 | s_2(\alpha) = \varepsilon(x) \}$, for $x \in G_0$ and $A = \{ A(x) \}_{x \in G_0}$. Then $A = (G_0, A)$ is a totally disconnected groupoid. Now we define a functor
\( \gamma : 2Gpd \to \text{Cmg} \) as an equivalence of categories such that \( \gamma(\mathcal{G}) = (\mathcal{A}, \mathcal{B}, \partial) \) is a crossed module over groupoids with \( \partial : \mathcal{A} \to \mathcal{B}, \ \partial(\alpha) = t_{\nu}(\alpha) \) and an action of groupoids such that \( f \cdot \alpha = 1_f \circ_h \alpha \circ_h 1_T \).

Clearly \( \partial(f \cdot \alpha) = f \circ \partial(\alpha) \circ \overline{f} \) and \( \partial(\alpha) \cdot \alpha_1 = \alpha \circ_h \alpha_1 \circ_h \overline{\alpha}_1 \), for \( f \in G_1(x, y) \) and \( \alpha, \alpha_1 \in A(x) \).

Let \( F = (F_0, F_1, F_2) \) be a morphism of 2-groupoids. Then \( \gamma(F) = (F_2|_{A'}, F_1, F_0) \) is a morphism of crossed modules over groupoids.

Now we define a functor \( \theta : \text{Cmg} \to 2Gpd \) which is an equivalence of categories. Let \( \mathcal{K} = (\mathcal{A}, \mathcal{B}, \partial) \) be a crossed module over groupoids \( \mathcal{A} = (X, A) \) and \( \mathcal{B} = (X, B) \). Then \( 2\)-groupoid \( \theta(\mathcal{K}) = (X, B, B \ltimes A) \) is a \( 2 \)-groupoid which is constructed as in the following way. The set of 2-morphisms is the semi-direct product \( B \ltimes A = \{(b, a) \mid b \in B, a \in A, s(a) = t(a) = t(b)\} \). If \( x \xrightarrow{b} y \xrightarrow{a} y \), then \((b, a)\) is a 2-morphism as follows:

where the horizontal composition of 2-morphisms is defined by
\[
(b_1, a_1) \circ_h (b_2, a_2) = (b_1 \circ_h b_2, a_1 \circ_h a_2)
\]
when \( y \xrightarrow{b_1} z \xrightarrow{a_1} z \) and the vertical composition of 2-morphisms is defined by
\[
\left( \partial(a) \circ b, a_2 \right) \circ_v (b, a) = (b, a_2 \circ a)
\]
when \( y \xrightarrow{a_2} y \). The source and the target maps are defined by \( s_h(b, a) = s(b), s_v(b, a) = b, t_h(b, a) = t(b), t_v(b, a) = \partial(a) \circ b \), respectively, the identity maps are defined by \( \varepsilon_h(x) = (1_x, 1_x), \varepsilon_v(b) = (b, 1_y) \), and the inversion maps are defined by \( \langle b, a \rangle^{-1} = (\partial(a) \circ b, a), \langle b, a \rangle^{-1} = (\overline{b}, b \circ \overline{a}) \).

Let \( \lambda = (\lambda_2, \lambda_1, \lambda_0) \) be a morphism of crossed modules over groupoids. Then
\[
\theta(\lambda) = (\lambda_0, \lambda_2, \lambda_2 \times \lambda_1)
\]
is a morphism of 2-groupoids.

A natural equivalence \( S : \theta \gamma \to 1_{2Gpd} \) is defined via the map \( S_{\mathcal{G}} : \theta \gamma(\mathcal{G}) \to \mathcal{G} \) which is defined to be identity on objects and on 1-morphisms, on 2-morphisms is defined by \( \alpha \mapsto (f, \alpha \circ_h 1_T) \). Clearly \( S_{\mathcal{G}} \) is an isomorphism and preserves compositions.

Now, given a crossed module \( \mathcal{K} = (\mathcal{A}, \mathcal{B}, \partial, \bullet) \) over groupoids, we define a natural equivalence \( T : 1_{\text{Cmg}} \to \gamma \theta \) by a map \( T_{\mathcal{K}} : \mathcal{K} \to \gamma \theta(\mathcal{K}) \) which is defined to be identity on objects and on \( B \), while on \( A \) is defined by \( a \mapsto (s(a), a) \).
3. Group-2-groupoids

In [1], a group-2-groupoid is defined as a group object in $2\text{Cat}$ using similar methods given in [6, 23]. In other words, a group-2-groupoid $G$ is a small 2-groupoid equipped with the following 2-functors satisfying group axioms, written out as commutative diagrams

1. $\mu: G \times G \to G$ called product,
2. $\text{inv}: G \to G$ called inverse and
3. $id: \{\ast\} \to G$ (where $\{\ast\}$ is a singleton) called unit or identity.

Then, the product of $x \xymatrix{ a \ar[d]^\alpha \ar[r]_b & y}$ and $x' \xymatrix{ a' \ar[d]^{\alpha'} \ar[r]_{b'} & y'}$ is written by $x \cdot x' \xymatrix{ a-b' \ar[r]_{\alpha \cdot \alpha'} & y \cdot y'}$, the inverse of $x \xymatrix{ a \ar[d]^\alpha \ar[r]_b & y}$ is $x^{-1} \xymatrix{ a^{-1} \ar[d]^{\alpha^{-1}} \ar[r]_{b^{-1}} & y^{-1}}$

where $id\{\ast\} = e \xymatrix{ 1 \ar[d]^e \ar[r]_1 & e}$. The condition 1 above gives us the following interchange rules

$$(a_1 \circ a) \cdot (a'_1 \circ a') = (a_1 \cdot a'_1) \circ (a \cdot a'),$$
$$(\delta \circ_{a_1} \alpha) \cdot (\delta' \circ_{a_1} \alpha') = (\delta \cdot \delta') \circ_{a_1} (\alpha \alpha'),$$
$$(\beta \circ_{a_1} \alpha) \cdot (\beta' \circ_{a_1} \alpha') = (\beta \cdot \beta') \circ_{a_1} (\alpha \cdot \alpha')$$

whenever compositions are defined. We can obtain from the condition 2 that $(a_1 \circ a)^{-1} = a_1^{-1} \circ a^{-1}$, $(\delta \circ_{a_1} \alpha)^{-1} = \delta^{-1} \circ_{a_1} \alpha^{-1}$, $(\beta \circ_{a_1} \alpha)^{-1} = \beta^{-1} \circ_{a_1} \alpha^{-1}$, $1_x^{-1} = 1_{x^{-1}}$, $1_{x^{-1}} = 1_{1_x^{-1}}$ and $1_{a^{-1}} = 1_{a^{-1}}$. Moreover, the structure of a group-2-groupoid $G = (G_0, G_1, G_2)$ contains compatible group-groupoids $G = (G_0, G_1)$, $G' = (G_0, G_2)$ and $G'' = (G_1, G_2)$ [1].

Equivalently we shall describe a group-2-groupoid as a 2-groupoid object in the category $\text{Gp}$ of groups. Let $C_0, C_1$ and $C_2$ be objects of a finitely complete category $C$. If $C_1 = (C_0, C_1, s, t, \varepsilon, \circ)$, $C_2 = (C_0, C_2, s_\delta, t_\delta, \varepsilon_\delta, \circ_\delta)$, and $C_3 = (C_1, C_2, s_\varepsilon, t_\varepsilon, \varepsilon_\varepsilon, \circ_\varepsilon)$ are internal categories in $C$ such that the following diagram commutes whenever the usual interchange rule satisfies between $\circ_h$ and $\circ_v$, then $(C_0, C_1, C_2)$ is called an internal 2-category in $C$.
Proposition 3.1. A 2-category object in \( \text{Gp} \) is a group-2-groupoid.

Proof. Let \( \mathcal{G} = (G_0, G_1, G_2) \) is a 2-category object in \( \text{Gp} \) and \( \mu_0, \mu_1, \mu_2 \) be multiplications of groups \( G_0, G_1, G_2 \), respectively. Then, we can define a multiplication \( \mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) as a 2-functor such that \( \mu = \mu_0 \) on objects, \( \mu = \mu_1 \) on 1-morphisms and \( \mu = \mu_2 \) on 2-morphisms. Similarly, we can define 2-functors \( \text{id}: 1 \to \mathcal{G} \) (where \( 1 \) is the terminal object of \( 2\text{Cat} \), i.e. the one-object discrete 2-category) which picks out an identity object, an identity 1-morphism and an identity 2-morphism and \( \text{inv}: \mathcal{G} \to \mathcal{G} \) picks out inverses for multiplications. Since \( a = 1_{s(a)}a^{-1}1_{t(a)} \) from \( 6 \) and \( \alpha' = 1_{s(v)}(\alpha^{-1}1_{t(v)}) \), \( \alpha' = 1_{s(v)}(\alpha^{-1}1_{t(v)}) \) from \( 1 \), \( \mathcal{G} \) is a 2-groupoid. Then, \( \mathcal{G} \) is a group object in \( 2\text{Cat} \) and so \( \mathcal{G} \) is a group-2-groupoid. \( \square \)

Example 3.2. Every group-groupoid can be thought as a group-2-groupoid in which all 2-morphisms are identities as follows:

\[
\begin{array}{ccc}
\begin{array}{c}
\xymatrix{a \ar[d]^{1_{s(a)}} & y} \\
\end{array}
& \cdot & \\
\begin{array}{c}
\xymatrix{a' \ar[d]^{1_{s(a')}} & y'} \\
\end{array}
\end{array}
= 
\begin{array}{ccc}
\begin{array}{c}
\xymatrix{a \cdot a' \ar[d]^{1_{s(a')}} & y \cdot y'} \\
\end{array}
\end{array}
\]

It is mentioned that a group-groupoid is a 2-category with a single object [4]. Then, we shall need a different viewpoint on group-groupoids as a special kind of group-2-groupoids:

Proposition 3.3. A group-2-groupoid with a single object is a group-groupoid in which both groups are necessarily abelian.

Proof. In this approach, the composition of 1-morphisms and the horizontal composition of 2-morphisms are defined by multiplications of groups as follows:

\[
\begin{array}{ccc}
\begin{array}{c}
\xymatrix{a \ar[d]^{1_{s(a)}} & b} \\
\end{array}
& \cdot & \\
\begin{array}{c}
\xymatrix{a' \ar[d]^{1_{s(a')}} & b'} \\
\end{array}
\end{array}
= 
\begin{array}{ccc}
\begin{array}{c}
\xymatrix{a' \star a \ar[d]^{1_{s(a') \cdot b'}} & b' \cdot b} \\
\end{array}
\end{array}
\]

It is proved in [23] that \( a' \star a = a' \cdot a = a \cdot a' \). Using similar way, we get

\[
\alpha' \star \alpha = (\alpha' \cdot 1_e) \star (1_e \cdot \alpha) = (\alpha' \star 1_e) \cdot (1_e \star \alpha) = \alpha' \cdot \alpha
\]

and

\[
\alpha' \cdot \alpha = (1_e \star \alpha') \cdot (\alpha \star 1_e) = (1_e \cdot \alpha) \star (\alpha' \cdot 1_e) = \alpha \cdot \alpha'.
\]

A third way to understand group-2-groupoids is to view them as double group-groupoids which are defined in [26] (see also [27]). Recall that a double category is a category object internal to \( \text{Cat} \). Hence the structure of a double category contains four different but compatible category structures as partially
shown in the following diagram

where $D^H_1$ and $D^V_1$ are called horizontal and vertical edge categories, respectively, and $D_2$ is called the set of squares. For further details, see [12, 14, 15, 20]. The structure of a 2-category may be regarded as a double category in which all vertical morphisms are identities (or $D_2$ and $D^H_1$ have the same objects) [12, 20]. Therefore, a group-2-groupoid is a special kind of an internal category in the category $\mathbf{GpGd}$ of group-groupoids.

4. Crossed modules over group-groupoids

In this section, we work on crossed modules over groupoids by replacing such groupoids with group-groupoids. Using the natural equivalence between crossed modules over groupoids and 2-groupoids given in [17], we will prove that there is a categorical equivalence between group-2-groupoids and crossed modules over group-groupoids.

**Definition 4.1.** Let $G = (X, G)$ and $H = (X, H)$ are group-groupoids over the same object set, $H$ be totally disconnected and $K = (H, G, \partial)$ be a crossed module over $G$ and $H$ such that $\partial$ is a homomorphism of group-groupoids and the following interchange rule holds:

$$(g \cdot h) \cdot (g' \cdot h') = (g \cdot g') \cdot (h \cdot h')$$

where $g, g' \in G, h, h' \in H$. Then $K$ is called a crossed module over group-groupoids.

A morphism of crossed modules over group-groupoids is a morphism of crossed modules of groupoids which preserves group structures. Then, we can construct the category of crossed modules over group-groupoids which we denote by $\mathbf{Cmg}^*$.

**Theorem 4.2.** The categories $\mathbf{Cmg}^*$ and $\mathbf{Gp2Gd}$ are equivalent.

**Proof.** The idea of the proof is to show that the functor of 2.2 restricts to an equivalence of categories. Let $A = (X, A)$ and $B = (X, B)$ are group-groupoids and $K = (A, B, \partial)$ is a crossed module over $A$ and $B$. Then $\theta(K) = (X, B, B \times A)$ is a group-2-groupoid via the process of the proof 2.2. The group multiplication of 2-morphisms in $\theta(K)$ is defined by

$$(b, a) \cdot (b', a') = (b \cdot b', a \cdot a').$$
We draw such pairs as

\[ \xrightarrow{\partial(a)\circ b} y \cdot x' \xleftarrow{\partial(a')\circ b'} y' = x \cdot x' \xrightarrow{\partial(a)\circ (b \cdot b')} y \cdot y' \]

Now we will verify that compositions and the group multiplication satisfy the interchange rule.

\[
\left[ (b_1, a_1) \circ_h (b, a) \right] \cdot \left[ (b_1', a_1') \circ_h (b', a') \right] = \left[ (b_1 \circ b, a_1 \circ (b_1 \bullet a)) \right] \cdot \left[ (b_1' \circ b', a_1' \circ (b_1' \bullet a')) \right] \\
= \left[ (b_1 \circ b) \cdot (b_1' \circ b'), (a_1 \circ (b_1 \bullet a) \cdot (a_1' \circ (b_1' \bullet a'))) \right] \\
= \left[ (b_1 \circ b_1') \circ (b \cdot b'), (a_1 \circ (b_1 \bullet a) \cdot (b_1' \bullet a')) \right] \\
= \left[ (b_1 \circ b_1') \circ (b \cdot b'), (a_1 \circ (b_1 \bullet a) \cdot (b_1' \bullet a')) \right] \\
= (b_1 \cdot b_1', a_1 \cdot a_1') \circ_h (b \cdot b', a \cdot a') \\
= \left[ (b_1, a_1) \right] \cdot \left[ (b_1', a_1') \right] \circ_h \left[ (b, a) \cdot (b', a') \right]
\]

and

\[
\left[ (\partial(a) \circ b, a_2) \circ_v (b, a) \right] \cdot \left[ (\partial(a') \circ b', a_2') \circ_v (b', a') \right] = \left[ (b, a_2 \circ a) \cdot (b', a_2' \circ a') \right] \\
= \left[ (b \circ b', (a_2 \circ a_2') \circ (a \circ a')) \right] \\
= \left[ (\partial(a) \circ a') \circ (b \circ b'), a_2 \circ a_2' \right] \circ_v (b \circ b', a \circ a') \\
= \left[ (\partial(a) \circ b, a_2) \cdot (\partial(a') \circ b', a_2') \right] \circ_v \left[ (b, a) \cdot (b', a') \right]
\]

whenever all above compositions are defined.

Now let \( \mathcal{G} = (G_0, G_1, G_2) \) be a group-2-groupoid. Then \( \gamma(\mathcal{G}) \) is a crossed module over groupoids internal to \( \text{Gp} \). We will verify that the interchange law holds:

\[(f \cdot a) \circ (f' \cdot a') = (1_f \circ_h \alpha \circ_h 1_f) \circ (1_{f'} \circ_h \alpha' \circ_h 1_{f'}) = 1_f \circ_h (\alpha \circ_h 1_f \cdot 1_{f'}) = (f \cdot f') \circ_h (\alpha \circ_h 1_f \cdot 1_{f'}) = (f \cdot f') \circ_h (\alpha \circ_h 1_f \cdot 1_{f'}) = S_\mathcal{G} (\alpha \cdot a') \]

Now we will show that \( S_\mathcal{G} \) preserves the group multiplication:

\[ S_\mathcal{G} (\alpha \cdot a') = (f \cdot f', (\alpha \cdot a') \circ_h 1_f \cdot 1_{f'}) \\
= (f \cdot f', (\alpha \circ_h 1_f \cdot 1_{f'}) \circ_h (\alpha' \circ_h 1_f \cdot 1_{f'}) \\
= (f \cdot f', (\alpha \circ_h 1_f \cdot 1_{f'}) \circ_h (\alpha' \circ_h 1_f \cdot 1_{f'}) \\
= (f, \alpha \circ_h 1_f \cdot 1_{f'}) \cdot (f', \alpha' \circ_h 1_f \cdot 1_{f'}) \\
= S_\mathcal{G} (\alpha) \cdot S_\mathcal{G} (\alpha') \\
\]

Other details are straightforward and so are omitted. \( \square \)
5. Group-2-groupoids as internal categories in $C_m$

A group-2-groupoid can be also thought as a special case of an internal category in the category $C_m$ of crossed modules (see, e.g., [25] and [26] for more details about internal categories in $C_m$). This idea comes from that the structure of a group-2-groupoid contains three compatible group-groupoid structures. Given a group-2-groupoid, we can extract crossed modules as follows:

$$\begin{array}{c}
G_2 \\
\downarrow s_v \\
G_1 \\
\downarrow t_v \\
G_0 \\
\downarrow s_h \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
Ker(s_h) \\
\downarrow s_v \\
Ker(s) \\
\downarrow t_v \\
G_0 \\
\downarrow s_h \\
\end{array}$$

Then, we obtain an internal groupoid in $C_m$

$$(Ker(s_h), G_0) \xrightarrow{e} (Ker(s), G_0)$$

where the structure maps are defined by $s = \langle s_v, 1 \rangle$, $t = \langle t_v, 1 \rangle$, $\epsilon = \langle \varepsilon_v, 1 \rangle$ as morphisms of crossed modules. Here $s, t, \epsilon$ are equivariant maps, since $s_v(x \cdot \alpha) = x \cdot s_v(\alpha)$, $t_v(x \cdot \alpha) = x \cdot t_v(\alpha)$ and $\varepsilon_v(x \cdot f) = x \cdot \varepsilon_v(f)$, for all $x \in G_0$ and $\alpha \in Ker(s_h)$. The actions of $G_0$ on $Ker(s_h)$ and on $Ker(s)$ are drawn in the following diagram:

$$xyx^{-1} := x \begin{array}{c}
\downarrow 1 \\
\downarrow 1_{[1_2]} \\
\end{array} x \cdot \begin{array}{c}
\downarrow 1 \\
\downarrow 1_{[1_2]} \\
\end{array} \cdot \epsilon \begin{array}{c}
\downarrow 1 \\
\downarrow 1_{[1_2]} \\
\end{array} y \begin{array}{c}
\downarrow 1 \\
\downarrow 1_{[1_2]} \\
\end{array} x^{-1} \begin{array}{c}
\downarrow 1 \\
\downarrow 1_{[1_2]} \\
\end{array} x^{-1}$$

We denote the category of such internal groupoids in $C_m$ by $IGC_m$. We know from [25, 26] that internal categories in the category $C_m$ of crossed modules are naturally equivalent to crossed squares which in turn should be viewed as a "crossed module of crossed modules". Hence an object of the category $IGC_m$ can be viewed as a special kind of crossed square.

Let $\mathcal{G} = (G_0, G_1, X, \partial_0, \partial_1)$ be an object of $IGC_m$. Then, the following diagram is commutative:

$$\begin{array}{c}
G_1 \\
\downarrow t \\
G_0 \\
\downarrow \partial_1 \\
\downarrow \partial_0 \\
X \\
\end{array}$$

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Let $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1), \mathfrak{G}' = (G'_0, G'_1, X', \partial'_0, \partial'_1)$ be objects of $\text{IGCm}$. If $(\lambda_1, \lambda_2)$ is an endomorphism of the group-groupoid $G = (G_0, G_1)$, and $(\lambda_2, \lambda_0)$ are morphisms of crossed modules $(G_0, X, \partial_0), (G_1, X, \partial_1)$, respectively, then $\lambda = (\lambda_2, \lambda_1, \lambda_0)$ is called a morphism of $\text{IGCm}$.

**Lemma 5.1.** Let $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1)$ be an object of $\text{IGCm}$. Then
\[ x \bullet (\beta \circ \alpha) = (x \bullet \beta) \circ (x \bullet \alpha) \]
for $x \in X, \alpha, \beta \in G_1$ where $s(\beta) = t(\alpha)$.

**Proof.** Let $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. We know from [6] that $\beta \circ \alpha = \beta \cdot 1_{b}^{-1} \cdot \alpha$. Then, we get
\[
\begin{align*}
  x \bullet (\beta \circ \alpha) &= x \bullet (\beta \cdot 1_{b}^{-1} \cdot \alpha) \\
  &= (x \bullet \beta) \cdot (x \bullet 1_{b})^{-1} \cdot (x \bullet \alpha) \\
  &= (x \bullet \beta) \cdot 1_{1_{b}}^{-1} \cdot (x \bullet \alpha) \\
  &= (x \bullet \beta) \circ (x \bullet \alpha)
\end{align*}
\]
\[ \square \]

**Example 5.2.** Every crossed module $K = (M, N, \partial)$ over groups is an object of $\text{IGCm}$ with the discrete groupoid of $M$ where $n \bullet 1_m = 1_{n \bullet m}$ and $\partial_1(1_m) = \partial(m)$.

**Theorem 5.3.** There is an equivalence between $\text{IGCm}$ and $\text{Gp2Gd}$.

**Proof.** A functor $\gamma: \text{Gp2Gd} \to \text{IGCm}$ is defined in the following way. Let $\mathcal{H} = (H_0, H_1, H_2)$ be a group-2-groupoid. Then $\gamma(\mathcal{H}) = (G_0, G_1, X, \partial_0, \partial_1)$ is an object of $\text{IGCm}$ where $G_0 = \text{Ker}(s), \ G_1 = \text{Ker}(s_h), \ X = H_0, \ \partial_0 = t |_{\text{Ker}(s)}, \ \partial_1 = t_h |_{\text{Ker}(s_h)}$.

\[
\mathcal{H} : \quad \begin{array}{c}
H_2 \\
\downarrow s_v \\
H_1 \\
\downarrow t_\varepsilon \\
H_0 \\
\downarrow t_h \\
\end{array} \quad \mapsto \quad \gamma(\mathcal{H}) : \quad \begin{array}{c}
G_1 \\
\downarrow s' \\
G_0 \\
\downarrow t' \\
X \\
\downarrow \partial_0 \\
\end{array}
\]

with actions $x \bullet f = 1_x \cdot f \cdot 1_x^{-1}$ and $x \bullet \alpha = 1_x \cdot \alpha \cdot 1_x^{-1}$, for $x \in X, f \in G_0, \alpha \in G_1$. Now we will verify that $s', t', \varepsilon'$ are equivariant maps.

$s'(x \bullet \alpha) = s'(1_x \cdot \alpha \cdot 1_x^{-1}) = s_v(1_x) \cdot s_v(\alpha) \cdot s_v(1_x^{-1}) = 1_x \cdot s_v(\alpha) \cdot 1_x^{-1} = x \bullet s'(\alpha),$ 
$t'(x \bullet \alpha) = t'(1_x \cdot \alpha \cdot 1_x^{-1}) = t_v(1_x) \cdot t_v(\alpha) \cdot t_v(1_x^{-1}) = 1_x \cdot t_v(\alpha) \cdot 1_x^{-1} = x \bullet t'(\alpha)$
Further remarks on group-2-groupoids

and

\[ \varepsilon'(x \cdot f) = \varepsilon'(1_x \cdot f \cdot 1_x^{-1}) = \varepsilon_v(1_x) \cdot \varepsilon_v(f) \cdot \varepsilon_v(1_x^{-1}) = 1_{1_x} \cdot \varepsilon_v(f) \cdot 1_{1_x}^{-1} = x \cdot \varepsilon'(f). \]

Let \( F = (F_0, F_1, F_2) \) be a morphism of group-2-groupoids. Then \( \gamma(F) = (F_2|_{Ker(s_0)}, F_1|_{Ker(s_0)}, F_0) \) is a morphism of \( \text{IGCm} \).

Next, we define a functor \( \theta : \text{IGCm} \rightarrow \text{Gp2Gd} \) is an equivalence of categories. Given an object \( \mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1) \) of \( \text{IGCm} \), we can obtain a group-2-groupoid \( \mathcal{H} = (H_0, H_1, H_2) \) where \( H_0 = X, H_1 = X \times G_0, H_2 = X \times G_1 \) as in the following way. Let \( a \xrightarrow{\alpha} b \) be a morphism of \( \mathfrak{G} \). Then pairs \( x \xrightarrow{(x,a)} \partial_0(a) \cdot x \) and \( x \xrightarrow{(x,b)} \partial_0(b) \cdot x \) are obtained as morphisms of the group-groupoid \( (H_0, H_1) \), and a pair \( x \xrightarrow{(x,a)} \partial_1(\alpha) \cdot x \) is obtained as a morphism of the group-groupoid \( (H_0, H_2) \). Since

\[ \partial_1(\alpha) \cdot x = \partial_0 s(\alpha) \cdot x = \partial_0(b) \cdot x, \quad \partial_1(\alpha) \cdot x = \partial_0 t(\alpha) \cdot x = \partial_0(b) \cdot x, \]

then \( (x, \alpha) \) can be considered as a 2-morphism as follows:

\[ x \xrightarrow{(x,a)} \partial_1(\alpha) \cdot x \]

Let \( a \xrightarrow{\alpha} b \xrightarrow{\beta} c \). Then, the vertical composition of \( (x, \alpha) \) and \( (x, \beta) \) is defined by

\[ (x, \beta) \circ_v (x, \alpha) = (x, \beta \circ \alpha) \]

where the source and the target maps are defined by \( s_v(x, \alpha) = (x, s(\alpha)) \) and \( t_v(x, \alpha) = (x, t(\alpha)) \), respectively, and the identity map is defined by \( \varepsilon_v(x, \alpha) = (x, 1_\alpha) \). Given morphisms \( a \xrightarrow{\alpha} b \) and \( a_1 \xrightarrow{\alpha_1} b_1 \), we obtain pairs \( (x, \alpha), (\partial_1(\alpha) \cdot x, \alpha_1) \) and we define their horizontal composite by

\[ (\partial_1(\alpha) \cdot x, \alpha_1) \circ_h (x, \alpha) = (x, \alpha_1 \cdot \alpha) \]

where the source and the target maps are defined by \( s_h(x, \alpha) = x, t_h(x, \alpha) = \partial_1(\alpha) \cdot x \), respectively, and the identity map is defined by \( \varepsilon_h(x) = (x, 1_c) \). Clearly the vertical composition and the horizontal composition satisfy the usual interchange rule. The product of \( (x, \alpha) \) and \( (x', \alpha') \) is written by

\[ (x, \alpha) \cdot (x', \alpha') = (x \cdot x', \alpha \cdot (x \cdot \alpha')) \]

for \( a \xrightarrow{\alpha} b \) and \( a' \xrightarrow{\alpha'} b' \).

If \( \lambda = (\lambda_2, \lambda_1, \lambda_0) \) is a morphism of \( \mathfrak{G} \), then \( \theta(\lambda) = (\lambda_0, \lambda_0 \times \lambda_1, \lambda_0 \times \lambda_2) \) is morphism of \( \theta(\mathfrak{G}) \).

A natural equivalence \( S : 1_{\text{Gp2Gd}} \rightarrow \theta \gamma \) is defined with a map \( S_G : \mathcal{G} \rightarrow \theta \gamma(\mathcal{G}) \) which is defined such that to be the identity on objects, \( S_G(f) = \)
(x, f \cdot 1_{x}^{-1}) and S_{\varphi}(\alpha) = (x, \alpha \cdot 1_{x}^{-1}) for f \in G_{1}, \alpha \in G_{2} where x = s(f) = s_{h}(\alpha). Clearly S_{\varphi} is an isomorphism and preserves the group operations and compositions as follows:

\[ S_{\varphi}(\alpha) \cdot S_{\varphi}(\alpha') = (x, \alpha \cdot 1_{x}^{-1}) \cdot (x', \alpha' \cdot 1_{x'}^{-1}) \]
\[ = (x \cdot x', \alpha \cdot 1_{x}^{-1} \cdot (x \cdot (\alpha' \cdot 1_{x'}^{-1}))) \]
\[ = (x \cdot x', \alpha \cdot 1_{x}^{-1} \cdot 1_{x} \cdot \alpha' \cdot 1_{x'}^{-1} \cdot 1_{x'}^{-1}) \]
\[ = (x \cdot x', \alpha \cdot \alpha' \cdot 1_{x'}^{-1}) \]
\[ = S_{\varphi}(\alpha \cdot \alpha') \]

where \( s(\alpha) = x, s(\alpha') = x' \),

\[ S_{\varphi}(\delta \circ_{h} \alpha) = S_{\varphi}(\delta \cdot 1_{x}^{-1} \cdot \alpha) = (x, \delta \cdot 1_{x}^{-1} \cdot \alpha \cdot 1_{x}^{-1}) = (y, \delta \cdot 1_{y}^{-1}) \circ_{h}(x, \alpha \cdot 1_{x}^{-1}) = S_{\varphi}(\delta) \circ_{h} S_{\varphi}(\alpha) \]

where \( t(\alpha) = s(\delta) = y \) and

\[ S_{\varphi}(\beta) \circ_{v} S_{\varphi}(\alpha) = (x, \beta \cdot 1_{x}^{-1}) \circ_{v} (x, \alpha \cdot 1_{x}^{-1}) \]
\[ = (x, (\beta \cdot 1_{x}^{-1}) \circ_{v} (\alpha \cdot 1_{x}^{-1})) \]
\[ = (x, (\beta \circ_{v} \alpha \cdot (1_{x}^{-1} \circ_{v} 1_{x}^{-1})) \]
\[ = (x, (\beta \circ_{v} \alpha) \cdot 1_{x}^{-1}) \]
\[ = S_{\varphi}(\beta \circ_{v} \alpha) \]

where \( s_{v}(\beta) = t_{v}(\alpha) \).

To define a natural equivalence \( T : 1_{IGCm} \rightarrow \gamma \theta \), a map \( T_{\varphi} \) is defined such that to be identity on \( X \), \( T_{\varphi}(a) = (e, a) \) for \( a \in G_{0} \) and \( T_{\varphi}(\alpha) = (e, \alpha) \) for \( \alpha \in G_{1} \). Obviously \( T_{\varphi} \) is an isomorphism and preserves the composition and the group multiplication as follows:

\[ T_{\varphi}(\beta \circ \alpha) = (e, \beta \circ \alpha) = (e, \beta) \circ (e, \alpha) = T_{\varphi}(\beta) \circ T_{\varphi}(\alpha) \]
\[ T_{\varphi}(\alpha) \cdot T_{\varphi}(\alpha') = (e, \alpha) \cdot (e, \alpha') = (e, \alpha \cdot (e \cdot \alpha')) = (e, \alpha \cdot \alpha') = T_{\varphi}(\alpha \cdot \alpha') \]

Other details are straightforward and so are omitted.

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Further remarks on group-2-groupoids

REFERENCES