

# Metric topology on the moduli space

JIALONG DENG

Mathematisches Institut, Georg-August-Universität, Göttingen, Germany  
([jialong.deng@mathematik.uni-goettingen.de](mailto:jialong.deng@mathematik.uni-goettingen.de))

*Communicated by S. Romaguera*

## ABSTRACT

---

*We define the smooth Lipschitz topology on the moduli space and show that each conformal class is dense in the moduli space endowed with Gromov-Hausdorff topology, which offers an answer to Tuschmann's question.*

---

2010 MSC: 54E35; 54F65; 51F99.

KEYWORDS: Gromov-Hausdorff topology;  $\varepsilon$ -topology; Lipschitz-topology; smooth Lipschitz-topology.

## 1. INTRODUCTION

Any smooth closed manifold  $M$  can be given a smooth Riemannian metric, and then one can ask: How many Riemannian metrics are there, and how many different geometries of this kind does the manifold actually allow? That means one wants to understand the space of Riemannian metrics on  $M$ , which is denoted by  $\mathcal{R}(M)$ , and the moduli space which is denoted by  $\mathcal{M}(M)$ . Here the moduli space is the quotient space of  $\mathcal{R}(M)$  by the action of diffeomorphism group of  $M$ . Those two questions originated from Riemann when he set up Riemannian geometry in the nineteenth century ([1]). Especially, the moduli space  $\mathcal{M}(M)$  is the superspace in physics (see [10], [4], [3]).

The  $C^{n,\alpha}$ -compact-open topology ( $n \in \mathbb{N}^+$  and  $\alpha \in \mathbb{R}^+$ ) is the most common consideration [9]. Tuschmann asked the following question in [8, Section 3, (8)]: *What can one say about the topology of moduli spaces under the Gromov-Hausdorff metric? What if one uses the Lipschitz topology?*

## 2. METRIC TOPOLOGY

Inspired by Tuschmann's questions, we will introduce four kinds of metric topology on the moduli space, and then discuss the relationship among them.

Let  $X$  and  $Y$  be metric spaces of finite diameter, then the Gromov-Hausdorff distance is defined as  $\rho_{GH}(X, Y) := \inf_Z \{d_H^Z(f(X), g(Y))\}$ , where  $d_H$  is Hausdorff metric and  $Z$  takes all metric spaces such that  $f$  (resp.  $g$ ) are isometric embeddings  $X$  (resp.  $Y$ ) into  $Z$  (see [5]). The Gromov-Hausdorff distance  $\rho_{GH}$  is a pseudo-metric in the collection of all compact metric spaces. Furthermore,  $\rho_{GH}(X, Y) = 0$  if and only if  $X$  is isometric to  $Y$ . For  $g_1$  and  $g_2$  in  $\mathcal{R}(M)$ , the Gromov-Hausdorff distance can be defined on it by  $\rho_{GH}(g_1, g_2) = \rho_{GH}((M, d_1), (M, d_2))$ , where  $d_1$  and  $d_2$  are induced metrics on  $M$  by  $g_1$  and  $g_2$ . Since  $M$  is closed, the Gromov-Hausdorff distance is well-defined on  $\mathcal{R}(M)$ . Moreover,  $\rho_{GH}(f_1^*g_1, f_2^*g_2) = \rho_{GH}(g_1, g_2)$ , where  $f_1$  and  $f_2$  are diffeomorphism of  $M$  and  $f_1^*g_1, f_2^*g_2$  are push-back metrics on  $M$ . Then one can define  $\rho_{GH}$  on  $\mathcal{M}(M)$  as above and then  $\rho_{GH}$  is a metric on  $\mathcal{M}(M)$ . Therefore,  $\mathcal{M}(M)$  can be endowed with the metric topology called GH-topology by the Gromov-Hausdorff metric  $\rho_{GH}$ .

**Definition 2.1** (Edwards [3]). A map  $f : X \rightarrow Y$  is called an  $\varepsilon$ -isometry between compact metric spaces  $X$  and  $Y$ , if  $|d_X(a, b) - d_Y(f(a), f(b))| \leq \varepsilon$  for all  $a, b \in X$ .

**Definition 2.2** ( $\varepsilon$ -distance). Assume  $g_1$  and  $g_2$  are in  $\mathcal{R}(M)$ , then we define the  $\varepsilon$ -distance by

$$\rho_\varepsilon(g_1, g_2) := \rho_\varepsilon(d_1, d_2) = \inf \{ \varepsilon \mid I_\varepsilon(d_1, d_2) \neq \emptyset \neq I_\varepsilon(d_2, d_1) \},$$

where  $I_\varepsilon(d_1, d_2)$  are the set of  $\varepsilon$ -isometries from  $(M, d_1)$  to  $(M, d_2)$ .

*Remark 2.3.* Note that

$$|\text{Diam}(d_1) - \text{Diam}(d_2)| \leq \rho_\varepsilon(d_1, d_2) \leq \max\{\text{Diam}(d_1), \text{Diam}(d_2)\},$$

where  $\text{Diam}(d_i)$  is the diameter of  $(M, g_i)$ ,  $i = 1, 2$ . Thus,  $\rho_\varepsilon$  is well-defined on  $\mathcal{R}(M)$ . Moreover,  $\rho_\varepsilon$  is the pseudo-metric and  $\rho_\varepsilon(g_1, g_2) = 0$  if and only if  $g_1$  is isometric to  $g_2$  on  $\mathcal{R}(M)$ .

Then  $\varepsilon$ -metric, which is also denoted by  $\rho_\varepsilon$ , can be defined on  $\mathcal{M}(M)$  as  $\rho_{GH}$ . Thus, it induces a metric topology on  $\mathcal{M}(M)$  called  $\varepsilon$ -topology. The conformal class dense theorem of the  $\varepsilon$ -topology on  $\mathcal{M}(M)$  was proved by Liu in [7, Corollary 2.2].

**Theorem 2.4** (Liu [7]). *Each conformal class is dense in  $\mathcal{M}(M)$  that is endowed with  $\varepsilon$ -topology.*

**Lemma 2.5.** *If  $\rho_{GH}(X, Y) \leq \varepsilon$ , then there is a  $2\varepsilon$ -isometric map  $f : X \rightarrow Y$ . If there is an  $\varepsilon$ -isometric map  $f : X \rightarrow Y$ , then  $\rho_{GH}(X, Y) \leq \frac{3}{2}\varepsilon$ .*

*Remark 2.6.* The lemma can be proved by using another definition of Gromov-Hausdorff metric, i.e.  $\rho_{GH}(X, Y) = \frac{1}{2} \inf_R \{ \text{dis}(R) \}$ , where the infimum is taken

over all correspondences  $R \subseteq X \times Y$ . A correspondence between two metric spaces  $X$  and  $Y$  is a subset  $R$  of  $X \times Y$  such that the projections  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  remain surjective when they are restricted to  $R$ .

**Corollary 2.7.**  *$\varepsilon$ -topology is equivalent to GH-topology on  $\mathcal{M}(M)$ .*

Thus, the conformal class dense theorem is also true on GH-topology. That means the GH-topology is coarse and a finer topology is needed to define on  $\mathcal{M}(M)$ .

Let  $X$  and  $Y$  be two compact metric spaces, the dilation of a Lipschitz map  $f : X \rightarrow Y$  is defined by

$$\text{Dil}(f) := \sup_{a,b \in X, a \neq b} \frac{d_Y(f(a), f(b))}{d_X(a, b)}.$$

If  $f^{-1}$  is also a Lipschitz map then it is called the bi-Lipschitz homeomorphism. The Lipschitz-distance  $\rho_L$  between  $X$  and  $Y$  is defined by

$$\rho_L(X, Y) := \inf_{f: X \rightarrow Y} \log\{\max\{\text{Dil}(f), \text{Dil}(f^{-1})\}\},$$

where the infimum is taken over all bi-Lipschitz homeomorphisms between  $X$  and  $Y$ . Then the Lipschitz-distance  $\rho_L$  can be defined on  $\mathcal{R}(M)$  as the definition of Gromov-Hausdorff distance on  $\mathcal{R}(M)$ .

Moreover,  $\rho_L$  is pseudo-metric on  $\mathcal{R}(M)$  and  $\rho_L(g_1, g_2) = 0$  if and only if  $g_1$  is isometric to  $g_2$  (see [2, Theorem 7.2.4]). Thus, it can induce a Lipschitz-metric  $\rho_L$  on  $\mathcal{M}(M)$ , and then  $\rho_L$  induces the Lipschitz-topology on  $\mathcal{M}(M)$  called L-topology. Furthermore, Lipschitz convergence implies Gromov-Hausdorff Convergence, where the convergence means Cauchy sequence convergence related to their metrics (see [6, Proposition 3.6]).

**Proposition 2.8.** *L-topology is finer than GH-topology on  $\mathcal{M}(M)$ .*

The GH-topology and L-topology on  $\mathcal{M}(M)$  only catch the metric information of the basic manifold and lose much essential information of the smooth structure. So it may be useful to modify the definition of L-topology on  $\mathcal{M}(M)$  to a finer topology on  $\mathcal{M}(M)$ .

For any homomorphism of metric space  $f : (X, d_X) \rightarrow (Y, d_Y)$ , the Lipschitz constant of  $f$  is defined by

$$L(f) := \inf\{k \geq 1 \mid \frac{d_X(x, y)}{k} \leq d_Y(f(x), f(y)) \leq kd_X(x, y), x, y \in X\}.$$

If the set is empty, then let  $L(f)$  be infinity.

**Lemma 2.9.** *Suppose that  $M$  and  $N$  are smooth closed Riemannian manifolds, then any diffeomorphism of  $M$  and  $N$  has bounded Lipschitz constant.*

*Remark 2.10.* The normal of tangent maps of diffeomorphism on the unit tangent bundle over closed manifold are uniform bounded, since the tangent maps are continuous and the total spaces of unit tangent bundle over compact manifold are compact.

For the composition of diffeomorphism  $f \circ g : M \rightarrow N \rightarrow W$ , we have  $L(f \circ g) \leq L(f) \cdot L(g)$  by direct computation.

**Definition 2.11.** Assume  $g_1$  and  $g_2$  are in  $\mathcal{R}(M)$ , we define

$$\rho_{SL}(g_1, g_2) = \rho_{SL}((M, d_1), (M, d_2)) := \inf\{\log L(f) \mid f \in \text{Diff}\},$$

where  $\text{Diff}$  is the diffeomorphism group of  $M$ .

**Lemma 2.12.**  $\rho_{SL}$  is a pseudo-metric on  $\mathcal{R}(M)$  and  $\rho_{SL}(g_1, g_2) = 0$  if and only if  $g_1$  is isometric to  $g_2$  on  $M$ .

*Remark 2.13.* If  $\rho_{SL}(d_1, d_2) = 0$ , then the isometry map between  $(M, d_1)$  and  $(M, d_2)$  can be constructed by using the closeness of  $M$  and the Arzela-Ascoli theorem.

Continuing the game, one can define the metric topology on  $\mathcal{M}(M)$  called SL-topology by the metric  $\rho_{SL}$ .

**Theorem 2.14.**  $SL\text{-topology} \preceq L\text{-topology} \preceq GH\text{-topology} \cong \varepsilon\text{-topology}$ .

Usually those four metrics are not complete metrics on  $\mathcal{M}(M)$ , so  $\mathcal{M}(M)$  is local compact topology spaces endowed with their induced metric topology in general. But if we restrict it to the subset of  $\mathcal{M}(M)$ , it may have some precompact propositions. For example, Gromov precompactness theorem and other convergence theorems on the moduli space [6, Chapter 5].

For the non-compact case, one can ask what is the right topology on  $\mathcal{R}^{\geq 0}(V)$  and  $\mathcal{M}^{\geq 0}(V)$ , where  $V$  is a non-compact manifold,  $\mathcal{R}^{\geq 0}(V)$  is the Riemannian metric with non-negative sectional curvature, and  $\mathcal{M}^{\geq 0}(V)$  is the moduli space of  $V$  with non-negative sectional curvature?

ACKNOWLEDGEMENTS. *I thank Xuchao Yao for useful discussions.*

## REFERENCES

- [1] N. A'Campo, L. Ji and A. Papadopoulos, On the early history of moduli and Teichmüller spaces, arXiv e-prints, page arXiv:1602.07208, Feb 2016.
- [2] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, volume 33, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.
- [3] D. A. Edwards, The structure of superspace, in: Studies in topology (Proc. Conf., Univ. North Carolina, Charlotte, N. C., 1974; dedicated to Math. Sect. Polish Acad. Sci.), pages 121–133, 1975.
- [4] A. E. Fischer, The theory of superspace, in: Relativity (Proc. Conf. Midwest, Cincinnati, Ohio, 1969), pages 303–357, 1970.
- [5] M. Gromov, Groups of polynomial growth and expanding maps, Inst. Hautes Études Sci. Publ. Math. 53 (1981), , 53–73.
- [6] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA (2007).

- [7] C.-H. Li, Quantum fluctuations, conformal deformations, and Gromov's topology — Wheeler, DeWitt, and Wilson meeting Gromov, arXiv e-prints, page arXiv:1509.03895, Sep 2015.
- [8] W. Tuschmann, Spaces and moduli spaces of Riemannian metrics, *Front. Math. China* 11, no. 5 (2016), 1335–1343.
- [9] W. Tuschmann and D. J. Wraith, *Moduli spaces of Riemannian metrics*, volume 46, Oberwolfach Seminars, Birkhäuser Verlag, Basel, 2015.
- [10] J. A. Wheeler, Superspace, in: *Analytic methods in mathematical physics (Sympos., Indiana Univ., Bloomington, Ind., 1968)*, pages 335–378, 1970.