Applied General Topology

Appl. Gen. Topol. 21, no. 2 (2020), 177-194
doi:10.4995/agt.2020.11369

Fixed points for fuzzy quasi-contractions in fuzzy metric spaces endowed with a graph

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Communicated by I. Altun

Abstract

In this paper, we introduce the notion of G-fuzzy $\mathcal{H}$-quasi-contractions using directed graphs in the setting of fuzzy metric spaces endowed with a graph and we show that this new type of contraction generalizes a large number of different types of contractions. Subsequently, we investigate some results concerning the existence of fixed points for this class of contractions under two different conditions in M-complete fuzzy metric spaces endowed with a graph. Our main results of the work significantly generalize many known comparable results in the existing literature. Examples are given to support the usability of our results and to show that they are improvements of some known ones.

2010 MSC: *54H25; 47H10; 05C40.*

Keywords: *fuzzy metric space; $(\widetilde{\mathrm{C}})$-graph; $\widetilde{G}$-fuzzy quasi-contraction; fixed point.*

1. Introduction and preliminaries

In attempt to model the real world problems, we have to deal with uncertainties and vagueness of the data, tools or conditions in the form of constraints. To deal with uncertainty, we need techniques other than classical ones wherein some specific logic is required. Fuzzy set theory is one of the uncertainty approaches wherein topological structures are basic tools to develop mathematical models compatible to concrete real life situations. Zadeh [21] considered the

Received 08 February 2019 – Accepted 24 June 2020

nature of uncertainty in the behaviour of systems possessing fuzzy nature by means of a fuzzy set.

In 1994, George and Veeramani [11] modified the concept of fuzzy metric space introduced by Kramosil and Michálek [15].

Definition 1.1 (George and Veeramani [11])**.** A fuzzy metric space is a triple $(X, M, \star)$ such that X is a nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

(FM1) $M(x, y, t) > 0$ for all $x, y \in X$ and each $t > 0$;
(FM2) $M(x, y, t) = 1$ for all $x, y \in X$ and each $t > 0$ if and only if $x = y$;
(FM3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and each $t > 0$;
(FM4) $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$ for all $x, y, z \in X$ and each $t, s > 0$;
(FM5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If we replace (FM4) by

(NA) $M(x, z, \max\{t, s\}) \geq M(x, y, t) \star M(y, z, s)$ for all $x, y, z \in X$ and each $t, s > 0$,

then the triple $(X, M, \star)$ is called a non-Archimedean fuzzy metric space. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.

Example 1.2 ([11])**.** Let (X, d) be a metric space. Then the triple $(X, M_d, \star)$ is a fuzzy metric space, where $a \star b = ab$ for all $a, b \in [0, 1]$ and $M_d(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and each $t > 0$. We call this M_d as the standard fuzzy metric induced by the metric d. Even if we define $a \star b = \min\{a, b\}$ for all $a, b \in [0, 1]$, then the triple $(X, M_d, \star)$ will be a fuzzy metric space.

Let $(X, M_d, \star)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \big\{y \in X \ : \ M(x, y, t) > 1 - r\big\}.$$

The collection $\{B(x, r, t) \ : \ x \in X, \ 0 < \epsilon < 1, \ t > 0\}$ is a neighbourhood system for the topology τ on X induced by the fuzzy metric M.

Lemma 1.3 ([9])**.** *Let $(X, M, \star)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

Definition 1.4 (George and Veeramani [11])**.** Let $(X, M, \star)$ be a fuzzy metric space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, denoted by $x_n \to x$ as $n \to \infty$, if and only if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.
(2) A sequence $\{x_n\}$ in X is a M-Cauchy sequence if and only if for all $\epsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$.

(3) An M-complete fuzzy metric space is a fuzzy metric space in which every M-Cauchy sequence is convergent.

Definition 1.5 ([10])**.** Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is called t-uniformly continuous if for all $r \in (0, 1)$, there exists $s \in (0, 1)$ such that

$$M(x, y, t) \geq 1 - s \qquad \text{implies} \qquad M(Tx, Ty, t) \geq 1 - r$$

for all $x, y \in X$ and each $t > 0$.

Remark 1.6. If T is t-uniformly continuous, then it is uniformly continuous for the uniformity generated by M and so it is continuous for the topology deduced from M. For the details concerning a uniform structure in a fuzzy metric space, the reader is directed to [10].

Fixed point theory is one of the most fruitful and effective tools in mathematics which has enormous applications within as well as outside the mathematics. The paper of Grabiec [9] started the investigations concerning fixed point theory in fuzzy metric spaces. Afterwards, Gregori and Sapena [10] introduced the notion of fuzzy contractive mappings and gave some fixed point results in fuzzy metric spaces.

Recently, Wardowski [20] introduced the following class of functions which will be used densely in the sequel.

Denote by $\mathcal{H}$ the family of all the mappings $\eta : (0, 1] \to [0, \infty)$ satisfying the following properties:

(H1) η transforms $(0, 1]$ onto $[0, \infty)$;
(H2) η is strictly decreasing (i.e. $s < t$ implies $\eta(s) > \eta(t)$ for all $t, s \in (0, 1]$).

It is worth mentioning that if $\eta \in \mathcal{H}$, then $\eta(1) = 0$ and η is continuous.

Theorem 1.7 (Wardowski [20])**.** *Let $(X, M, \star)$ be an M-complete fuzzy metric space and suppose that $T : X \to X$ be a self-mapping satisfying*

$$\eta\big(M(Tx, Ty, t)\big) \leq k\eta\big(M(x, y, t)\big) \tag{1.1}$$

for all $x, y \in X$ and each $t > 0$, where $k \in (0, 1)$. Assume also that the following assertions hold:

(a) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$ *for all $x \in X$, $k \in \mathbb{N}$ and any sequence $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$;*
(b) $r \star s > 0$ *implies $\eta(r \star s) \leq \eta(r) + \eta(s)$ for all $r, s \in \big\{M(x, Tx, t) : x \in X, \, t > 0\big\}$;*
(c) $\big\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\big\}$ *is bounded for all $x \in X$ and any sequence $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.*

Then T has a unique fixed point in $x^* \in X$ and for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

By considering a mapping $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t} - 1$ where $t \in (0, 1]$, the fuzzy contraction condition (1.1) reduces to the class of fuzzy contractive mappings introduced by Gregori and Sapena [10].

On the other hand, the most important graph theory approach to metric fixed point theory introduced so far is attributed to Jachymski [13]. In this new approach, the underlying metric space is equipped with a directed graph and the Banach contraction is formulated in a graph language. Using this simple but very interesting idea, Jachymski generalized several well known versions of Banach contraction principle in metric spaces simultaneously and from various aspects. We commence by reviewing some basic notions in graph theory which will be used throughout this paper. The readers interested in this topic are referred to [5, 8, 19] and references cited therein.

In an arbitrary (not necessarily simple) graph G, a link is an edge of G with distinct ends and a loop is an edge of G with identical ends. Two or more links of G with the same pairs of ends are called parallel edges of G.

Let $(X, M, \star)$ be a fuzzy metric space and $\Delta(X)$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of the vertices of G coincides with X, i.e. $V(G) = X$, and the set $E(G)$ of the edges of G contains all loops, i.e. $E(G) \supseteq \Delta(X)$ (note that in general, G can have uncountably many vertices). Suppose further that G has no parallel edges. In this case, the graph G can be simply denoted by the ordered pair $G = (V(G), E(G)) = (X, E(G))$. If G is such a graph, then it is said that the fuzzy metric space $(X, M, \star)$ is endowed with the graph G.

By the notation G^{-1}, it is meant the conversion of G as usual, i.e. a directed graph obtained from G by reversing the directions of the edges of G, and by the notation $\widetilde{G}$, it is always meant the undirected graph obtained from G by ignoring the directions of the edges of G. Thus, it is clear that $V(G^{-1}) = V(\widetilde{G}) = V(G) = X$ and so

$$E(G^{-1}) = \big\{(x, y) \in X \times X \,:\, (y, x) \in E(G)\big\} \quad \text{and} \quad E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

It should be remarked that if both (x, y) and (y, x) belong to $E(G)$, then we will face with parallel edges in the graph $\widetilde{G}$. To avoid this problem, we delete either the edge (x, y) or the edge (y, x) (but not both of them) from G and consider the graph $\widetilde{G}$ obtained from the remaining graph.

If $(X, \preceq)$ is a partially ordered set, then by comparable elements of $(X, \preceq)$, it is meant two elements $x, y \in X$ satisfying either $x \preceq y$ or $y \preceq x$, and a mapping $T : X \to X$ is called order-preserving whenever $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

In 1974, Ćirić [7] introduced quasi-contractions in metric spaces and gave an example to show that this new contraction is a real generalization of some well known linear contraction. The main purpose of the present work is to formulate a G-fuzzy $\mathcal{H}$-quasi-contraction which generalizes a large number of contractions in fuzzy metric spaces endowed with a graph. We then investigate some sufficient conditions which ensure the existence of fixed points for such mappings on M-complete fuzzy metric spaces in the sense of George and Veeramani endowed with a graph. The obtained results generalize many known results in the recent literature. Some examples are given which illustrate the value of the obtained results in comparison to some of the existing ones in literature.

2. Main results

Suppose that $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be an arbitrary mapping. Throughout this section, we use the letter C_T to denote the set of all points $x \in X$ such that $(T^m x, T^n x) \in E(\widetilde{G})$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Remark 2.1. Let $(\mathbb{R}, d)$ be the usual (Euclidean) metric space of all real numbers and $(\mathbb{R}, M_d, \star)$ be the standard fuzzy metric space induced by d. Consider a graph G given by $V(G) = \mathbb{R}$ and $E(G) = \{(x, x) : x \in \mathbb{R}\}$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by the rule $Tx = x + 2$ for all $x \in \mathbb{R}$. Now one can see easily that $C_T = \varnothing$.

Given $x \in X$ and $n \in \mathbb{N} \cup \{0\}$, the n-th orbit of x under T is denoted by $O(x; n)$, i.e.

$$O(x; n) = \big\{x, Tx, \dots, T^n x\big\}.$$

If A is a subset of X, then by $\delta_t(A)$, it is meant the diameter of A in X, i.e.

$$\delta_t(A) = \sup\big\{\eta\big(M(x, y, t)\big) : x, y \in A\big\}.$$

Motivated by Aleomraninejad et al. [1], we say that G is a $(\widetilde{\mathrm{C}})$-graph whenever the triple (X, d, G) has the following property:

> If $\{x_n\}$ is a sequence in (X, d, G) such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(\widetilde{G})$ for all $k \in \mathbb{N}$.

Now, we are ready to introduce the concept of G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ in fuzzy metric spaces endowed with a graph which is inspired by [2, Definition 2.2] and [13, Definition 2.1].

Definition 2.2. Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a mapping. We say T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ if

(FQ1) T preserves the edges of G, i.e. $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(FQ2) there exists $\lambda \in (0, 1)$ such that

$$\eta\big(M(Tx, Ty, t)\big) \le \lambda \max\big\{\eta\big(M(x, y, t)\big), \eta\big(M(x, Tx, t)\big), \eta\big(M(y, Ty, t)\big)$$
$$\eta\big(M(x, Ty, t)\big), \eta\big(M(y, Tx, t)\big)\big\}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and each $t > 0$.

If T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$, then we call λ in (FQ2) a quasi-contractive constant of T.

We now give some examples of G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ in fuzzy metric spaces endowed with a graph.

Example 2.3. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and x_0 be a point in X. It is elementary to check that the constant mapping $x \overset{T}{\mapsto} x_0$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with arbitrary quasi-contractive constant $\lambda \in (0, 1)$ since $E(G)$ contains all the loops. So the cardinality of the set of all G-fuzzy $\mathcal{H}$-quasi-contractions defined on a fuzzy metric space $(X, M, \star)$ endowed with a graph G is no less than the cardinality of X.

Example 2.4. Suppose that $(X, M, \star)$ is a fuzzy metric space and $T : X \to X$ is a fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ in the sense that there exists $\lambda \in (0, 1)$ such that

$$\begin{aligned} \eta\big(M(Tx, Ty, t)\big) \leq \lambda \max\big\{&\eta\big(M(x, y, t)\big), \eta\big(M(x, Tx, t)\big), \eta\big(M(y, Ty, t)\big), \\ &\eta\big(M(x, Ty, t)\big), \eta\big(M(y, Tx, t)\big)\big\} \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and any $t > 0$. Define a graph G_0 by $V(G_0) = X$ and $E(G_0) = X \times X$, i.e. G_0 is the complete graph whose vertex set coincides with X. Obviously, T preserves the edges of G_0 and (2.1) guarantees that T satisfies (FQ2) for the complete graph G_0. Thus, T is a G_0-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with the quasi-contractive constant $\lambda \in (0, 1)$. Hence, G_0-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ on fuzzy metric spaces endowed with the graph G_0 are precisely the fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ on fuzzy metric spaces. Therefore, G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ are a generalization of fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ from fuzzy metric spaces to fuzzy metric spaces endowed with a graph. As stated before, the concept of quasi-contractions in metric spaces initiated by Ćirić [7] in 1974. Moreover, Rhoades [18] showed that Ćirić's contractive condition is one of the most general contractive definitions in metric spaces and includes a large number of different types of contractions.

Example 2.5. Let $(X, \preceq)$ be a partially ordered set and $(X, M, \star)$ be a fuzzy metric space. Consider the poset graphs G_1 and G_2 by

$$V(G_1) = X \qquad \text{and} \qquad E(G_1) = \big\{(x, y) \in X \times X \,:\, x \preceq y\big\},$$

and

$$V(G_2) = X \qquad \text{and} \qquad E(G_2) = \big\{(x, y) \in X \times X \,:\, x \preceq y \,\vee\, y \preceq x\big\}.$$

A mapping $T : X \to X$ preserves the edges of G_1 if and only if T is order-preserving, and T satisfies (FQ2) for the graph G_1 if and only if there exists $\lambda \in (0, 1)$ such that

$$\begin{aligned} \eta\big(M(Tx, Ty, t)\big) \leq \lambda \max\big\{&\eta\big(M(x, y, t)\big), \eta\big(M(x, Tx, t)\big), \eta\big(M(y, Ty, t)\big), \\ &\eta\big(M(x, Ty, t)\big), \eta\big(M(y, Tx, t)\big)\big\} \end{aligned} \tag{2.2}$$

for all comparable elements $x, y \in X$ and any $t > 0$, where $\eta \in \mathcal{H}$. Moreover, T preserves the edges of G_2 if and only if T maps comparable elements of $(X, \preceq)$ onto comparable elements, and T satisfies (FQ2) for the graph G_2 if and only if (2.2) holds for all comparable elements $x, y \in X$ and any $t > 0$.

Hence, if T is a G_1-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$, then T is a G_2-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Therefore, G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ are a generalization of ordered fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ from fuzzy metric spaces equipped with a partial order to fuzzy metric spaces endowed with a graph.

From now on, we assume that the graphs G_0, G_1 and G_2 are as defined in Examples 2.4 and 2.5.

Remark 2.6. In the definitions of $(\widetilde{\mathrm{C}})$-graph and the set C_T, let's set G the special graphs G_0, G_1 and G_2. Then we obtain the following special cases:

- The set C_T related to the complete graph G_0 coincides with X and G_0 is a $(\widetilde{\mathrm{C}})$-graph.
- If $\preceq$ is a partial order on X, then the set C_T related to the graph G_1 (and also G_2) consists of all points $x \in X$ whose every two iterates under T are comparable elements of X. Moreover, G_1 (and also G_2) is a $(\widetilde{\mathrm{C}})$-graph whenever the quadruple $(X, M, \star, \preceq)$ has the following property:
  - **$(*)$:** If $\{x_n\}$ is a sequence in $(X, M, \star)$ converging to a point $x \in X$ whose successive terms are pairwise comparable elements of $(X, \preceq)$, then there exists a subsequence of $\{x_n\}$ whose terms and x are comparable elements of $(X, \preceq)$.

Example 2.7. Let $X = [0, 1]$ and $\star$ be the usual product. Then $(X, M, \star)$ is a fuzzy metric space, where

$$M(x, y, t) = \left(\frac{t}{t+1}\right)^{|x-y|}$$

for all $x, y \in X$ and each $t > 0$. Define a self-mapping $T : X \to X$ by the formula

$$Tx = \begin{cases} \frac{1}{9}, & x = 0, \\ \frac{1}{3}, & 0 < x \leq 1. \end{cases}$$

We show that T is not a G_0-fuzzy $\mathcal{H}$-Banach contraction with respect to $\eta \in \mathcal{H}$ on X. Arguing by contradiction, we suppose that there exists $\eta \in \mathcal{H}$ such that

$$\eta\big(M(Tx, Ty, t)\big) \leq \lambda \eta\big(M(x, y, t)\big)$$

for all $x, y \in X$ and each $t > 0$, where $\lambda \in (0, 1)$ is a constant. Now, by taking the points $x = 0$, $0 < y \leq 1$ and $t = 1$ in the above inequality, we get $\eta((\frac{1}{2})^{\frac{2}{9}}) \leq \lambda \eta((\frac{1}{2})^{y})$ and so

$$\eta\big(\big(\frac{1}{2}\big)^{\frac{2}{9}}\big) \leq \lambda \lim_{y \to 0^+} \eta\big(\big(\frac{1}{2}\big)^{y}\big) = \lambda \eta(1) = 0,$$

which gives a contradiction.

On the other hand, from the equality

$$\ln M(\frac{1}{9}, \frac{1}{3}, t) = \frac{2}{3} \ln M(0, \frac{1}{3}, t) \qquad \text{for all } t > 0,$$

it immediately follows that for all $x, y \in X$ and each $t > 0$,

$$\eta\big(M(Tx,Ty,t)\big) \leq \frac{2}{3} \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big), \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\},$$

by considering a mapping $\eta \in \mathcal{H}$ of the form $\eta(s) = \ln(\frac{1}{s})$ for $s \in (0,1]$. Therefore, T is a G_0-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with the quasi-contractive constant $\lambda = \frac{2}{3}$.

Remark 2.8. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ is a G-fuzzy $\mathcal{H}$-Banach contraction with respect to $\eta \in \mathcal{H}$ in the sense that T preserves the edges of G and there exists $\alpha \in (0,1)$ such that

$$\eta\big(M(Tx,Ty,t)\big) \leq \alpha\eta\big(M(x,y,t)\big)$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and any $t > 0$. If $(x, y) \in E(G)$, then

$$\begin{aligned}\eta\big(M(Tx,Ty,t)\big) &\leq \alpha\eta\big(M(x,y,t)\big)\\ &\leq \alpha \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big),\\ &\qquad \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\}.\end{aligned}$$

Therefore, T satisfies (FQ2) and so T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Hence every G-fuzzy $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$.

Remark 2.9. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ is a G-fuzzy $\mathcal{H}$-Kannan contraction with respect to $\eta \in \mathcal{H}$ in the sense that T preserves the edges of G and there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\eta\big(M(Tx,Ty,t)\big) \leq \alpha\big(\eta\big(M(x,Tx,t)\big) + \eta\big(M(y,Ty,t)\big)\big)$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and any $t > 0$ (see [14] for the definition in metric spaces). If $(x, y) \in E(G)$, then

$$\begin{aligned}\eta\big(M(Tx,Ty,t)\big) &\leq \alpha\big(\eta\big(M(x,Tx,t)\big) + \eta\big(M(y,Ty,t)\big)\big)\\ &\leq 2\alpha \max\big\{\eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big)\big\}\\ &\leq 2\alpha \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big),\\ &\qquad \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\}.\end{aligned}$$

Therefore, T satisfies (FQ2) and so T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Hence every G-fuzzy $\mathcal{H}$-Kannan contraction with respect to $\eta \in \mathcal{H}$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$.

Remark 2.10. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ is a G-fuzzy $\mathcal{H}$-Chatterjea contraction with respect to

$\eta \in \mathcal{H}$ in the sense that T preserves the edges of G and there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\eta\big(M(Tx,Ty,t)\big) \leq \alpha\big(\eta\big(M(x,Ty,t)\big) + \eta\big(M(y,Tx,t)\big)\big)$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and any $t > 0$ (see [6] for the definition in metric spaces). If $(x,y) \in E(G)$, then an argument similar to that appeared in Remark 2.9 establishes that

$$\eta\big(M(Tx,Ty,t)\big) \leq 2\alpha \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big), \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\}.$$

Therefore, T satisfies (FQ2) and so T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Hence every G-fuzzy $\mathcal{H}$-Chatterjea contraction with respect to $\eta \in \mathcal{H}$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$.

Remark 2.11. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ is a G-fuzzy $\mathcal{H}$-Ćirić-Reich-Rus contraction with respect to $\eta \in \mathcal{H}$ in the sense that T preserves the edges of G and there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that

$$\eta\big(M(Tx,Ty,t)\big) \leq \alpha\eta\big(M(x,y,t)\big) + \beta\eta\big(M(x,Tx,t)\big) + \gamma\eta\big(M(y,Ty,t)\big)$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and any $t > 0$ (see [17] for the definition in metric spaces). If $(x,y) \in E(G)$, then an argument similar to that appeared in Remark 2.9 establishes that

$$\eta\big(M(Tx,Ty,t)\big) \leq (\alpha+\beta+\gamma) \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big), \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\}.$$

Therefore, T satisfies (FQ2) and so T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Hence every G-fuzzy $\mathcal{H}$-Ćirić-Reich-Rus contraction with respect to $\eta \in \mathcal{H}$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$.

Remark 2.12. Suppose that $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ is a G-fuzzy $\mathcal{H}$-Hardy-Rogers contraction with respect to $\eta \in \mathcal{H}$ in the sense that T preserves the edges of G and there exist $\alpha, \beta, \gamma, \delta, \theta \geq 0$ with $\alpha + \beta + \gamma + \delta + \theta < 1$ such that

$$\eta\big(M(Tx,Ty,t)\big) \leq \alpha\eta\big(M(x,y,t)\big) + \beta\eta\big(M(x,Tx,t)\big) + \gamma\eta\big(M(y,Ty,t)\big) + \delta\eta\big(M(x,Ty,t)\big) + \theta\eta\big(M(y,Tx,t)\big)$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and any $t > 0$ (see [12] for the definition in metric spaces). If $(x,y) \in E(G)$, then an argument similar to that appeared in Remark 2.9 establishes that

$$\eta\big(M(Tx,Ty,t)\big) \leq (\alpha+\beta+\gamma+\delta+\theta) \max\big\{\eta\big(M(x,y,t)\big), \eta\big(M(x,Tx,t)\big), \eta\big(M(y,Ty,t)\big), \eta\big(M(x,Ty,t)\big), \eta\big(M(y,Tx,t)\big)\big\}.$$

Therefore, T satisfies (FQ2) and so T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Hence every G-fuzzy $\mathcal{H}$-Hardy-Rogers contraction with respect to $\eta \in \mathcal{H}$ is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$.

Note that every fuzzy $\mathcal{H}$-contractive mapping (1.1) due to Wardowski [20] is t-uniformly continuous. In the next example, we see that a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ need not be even continuous.

Example 2.13. Let $(\mathbb{R}^+, d)$ be the usual (Euclidean) metric space of all non-negative real numbers and $(\mathbb{R}^+, M_d, \star)$ be the standard fuzzy metric space induced by d. Consider a graph G given by $V(G) = \mathbb{R}^+$ and

$$E(G) = \Delta \cup \big\{(x, y) \in X \times X \ : \ x, y \in \mathbb{Q} \cap \mathbb{R}^+ \text{ with } x \leq y\big\},$$

where $\mathbb{Q}$ is the set of all rational numbers. Define a mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ by the rule

$$Tx = \begin{cases} \frac{x}{2}, & x \in \mathbb{R}^+ \cap \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Then T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with the quasi-contractive constant $\lambda = \frac{1}{2}$. Obviously, T is only continuous at zero. In particular, T is not continuous on the whole $\mathbb{R}^+$.

The following proposition is an immediate consequence of the definition of G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ and gives a simple procedure to construct new G-fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ from older ones.

Proposition 2.14. *Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a mapping.*

1. *If T preserves the edges of G, then T preserves the edges of G^{-1} (resp. $\widetilde{G}$);*
2. *If T satisfies* (FQ2) *for the graph G, then T satisfies* (FQ2) *for the graph G^{-1} (resp. $\widetilde{G}$);*
3. *If T is a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with a quasi-contractive constant $\lambda \in (0, 1)$, then T is a G^{-1}-fuzzy $\mathcal{H}$-quasi-contraction (resp. $\widetilde{G}$-fuzzy $\mathcal{H}$-quasi-contraction) with respect to $\eta \in \mathcal{H}$ with a quasi-contractive constant λ.*

To prove the existence of a fixed point for a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$, we make use of the following useful lemmas.

Lemma 2.15. *Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with a quasi-contractive constant $\lambda \in (0, 1)$. Then*

$$\eta\big(M(T^i x, T^j x, t)\big) \leq \lambda \delta_t\big(O(x; n)\big) \qquad i, j = 1, \ldots, n$$

for all $x \in C_T$ and each $t > 0$ and any $n \in \mathbb{N}$.

Proof. Suppose that $x \in C_T$ and $n \in \mathbb{N}$ be given. If i and j are arbitrary positive integers no more than n, then $(T^{i-1}x, T^{j-1}x) \in E(\widetilde{G})$. According to Proposition 2.14, T is also a $\widetilde{G}$-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$

with a quasi-contractive constant $\lambda \in (0,1)$. In particular, T satisfies (FQ2) for the graph $\widetilde{G}$. Hence, we have

$$\begin{aligned}\eta\big(M(T^ix,T^jx,t)\big) &= \eta\big(M(TT^{i-1}x,TT^{j-1}x,t)\big)\\ &\leq \lambda \max\big\{\eta\big(M(T^{i-1}x,T^{j-1}x,t)\big),\eta\big(M(T^{i-1}x,T^ix,t)\big),\\ &\quad \eta\big(M(T^{j-1}x,T^jx,t)\big),\eta\big(M(T^{i-1}x,T^jx,t)\big),\\ &\quad \eta\big(M(T^{j-1}x,T^ix,t)\big)\big\}\\ &\leq \lambda\delta_t\big(O(x;n)\big)\end{aligned}$$

for all $t > 0$. □

Example 2.16. Consider the set $\mathbb{R}$ of real numbers with the usual (Euclidean) metric and the standard fuzzy metric space $(\mathbb{R}, M, \star)$. Let G_0 be a the complete graph and define a mapping $T : \mathbb{R} \to \mathbb{R}$ as $Tx = \frac{x}{2}$ for all $x \in \mathbb{R}$. Then T is a G_0-quasi-contraction with respect to $\eta \in \mathcal{H}$ with respect to $\eta \in \mathcal{H}$ with a quasi-contractive constant $\lambda = \frac{1}{2}$. Meanwhile, $T^nx = \frac{x}{2^n}$ and $\delta_t(O(x;n)) = \frac{|x|}{t}(1-\frac{1}{2^n})$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N} \cup \{0\}$. Now, let x_0 be a positive real numbers. Take a mapping $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t} - 1$ for $t \in (0,1]$ and put $n = 2$, $i = 0$ and $j = 1$ in Lemma 2.14. Hence, we have

$$\eta\big(M(x,Tx,t)\big) = \frac{|x_0 - Tx_0|}{t} = \frac{x_0}{2t} > \frac{x_0}{2t}\Big(1-\frac{1}{2^2}\Big) = \lambda\delta_t\big(O(x_0;2)\big).$$

Lemma 2.17. *Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$. Then for all $x \in C_T$ and each $n \in \mathbb{N}$, there exists a positive integer k no more than n such that*

$$\delta_t\big(O(x;n)\big) = \eta\big(M(x,T^kx,t)\big)$$

for all $t > 0$.

Proof. Suppose that $x \in C_T$ and $n \in \mathbb{N}$ be given. On the one hand, if $\delta_t\big(O(x;n)\big) = 0$, then $O(x;n)$ is singleton. In particular, x is a fixed point for T and $\eta\big(M(T^ix,T^jx,t)\big) = 0$ for all $i,j = 0,1,\ldots,n$ and each $t > 0$. Hence, the assertion holds trivially for any positive integer k no more than n.

On the other hand, if since $O(x;n)$ is a finite set, it follows that there exist distinct nonnegative integers i and j no more that n such that $\delta_t\big(O(x;n)\big) = \eta\big(M(T^ix,T^jx,t)\big)$ for all $t > 0$. If both the integers i and j are assumed to be positive, then from Lemma 2.15, we have

$$\delta_t\big(O(x;n)\big) = \eta\big(M(T^ix,T^jx,t)\big) \leq \lambda\delta_t\big(O(x;n)\big)$$

for all $t > 0$, where $\lambda \in (0,1)$ is a quasi-contraction constant of T, which is a contradiction. Therefore, either i or j must be zero. □

Remark 2.18. Combining Lemmas 2.15 and 2.17, one can easily obtain that if $(X, M, \star)$ is a fuzzy metric space endowed with a graph G and $T : X \to X$ be a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with a quasi-contraction constant λ, then for all $x \in C_T$ and each $n \in \mathbb{N}$, there exists a positive integer k no more than n such that

$$\eta\big(M(T^i x, T^j x, t)\big) \leq \lambda \delta_t\big(O(x;n)\big) = \lambda \eta\big(M(x, T^k x, t)\big) \qquad i,j = 1, \dots, n$$

for all $t > 0$.

Lemma 2.19. *Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ such that*

(i) *$\tau \geq r \star s$ implies $\eta(\tau) \leq \eta(r) + \eta(s)$ for all $r, s, \tau \in \big\{M(T^i x, T^j x, t) : x \in X,\ t > 0,\ i, j \in \mathbb{N}\big\}$;*
(ii) *$\big\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\big\}$ is bounded for all $x \in X$ and any sequence $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.*

Then the sequence $\{T^n x\}$ is Cauchy for all $x \in C_T$.

Proof. Suppose that $x \in C_T$ and $n \in \mathbb{N} \cup \{0\}$ be given. If $n = 0$, then there remains nothing to prove since $\delta_t\big(O(x;0)\big) = 0$. Otherwise, from Lemma 2.17, there exists a positive integer k no more than n such that

$$\delta_t\big(O(x;n)\big) = \eta\big(M(x, T^k x, t)\big) \qquad \text{for all } t > 0. \tag{2.3}$$

Now, we can choose a strictly decreasing sequence of positive numbers $\{a_i\}$ with $\sum_{i=1}^{\infty} a_i = 1$ and so thanks to (i) and (2.3), we obtain

$$\begin{aligned} \delta_t\big(O(x;n)\big) &= \eta\big(M(x, T^k x, t)\big) \\ &= \eta\big(M(x, T^k x, \sum_{i=1}^{\infty} a_i t)\big) \\ &\leq \eta\big(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)\big) + \eta\big(M(Tx, T^k x, \sum_{i=1}^{j} a_i t)\big) \end{aligned}$$

for all $t > 0$ and any j. Hence, by substituting $i = 1$ and $j = k$ in Lemma 2.15, we get

$$\begin{aligned} \delta_t\big(O(x;n)\big) &\leq \limsup_{j \to \infty} \eta\big(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)\big) + \eta\big(M(Tx, T^k x, t)\big) \\ &\leq \limsup_{j \to \infty} \eta\big(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)\big) + \lambda \delta_t\big(O(x;n)\big), \end{aligned}$$

from which it follows that

$$\delta_t\big(O(x;n)\big) \leq \frac{1}{1-\lambda} \limsup_{j \to \infty} \eta\big(M(x, Tx, \sum_{i=j+1}^{\infty} a_i t)\big). \tag{2.4}$$

If $m,n\in\mathbb{N}$ with $m\geq n\geq 2$, since $T^{n-1}x\in C_T$, then by putting $i=m-n+1$ and $j=1$ in Lemma 2.15, we obtain

$$\begin{aligned}\eta\big(M(T^mx,T^nx,t)\big)&=\eta\big(M(T^{m-n+1}T^{n-1}x,TT^{n-1}x,t)\big)\\&\leq\lambda\delta_t\big(O(T^{n-1}x;m-n+1)\big),\end{aligned}\tag{2.5}$$

for all $t>0$, where $\lambda\in(0,1)$ is a quasi-contractive constant of T. Moreover, due to (2.3), there exists a positive integer k no more than $m-n+1$ such that

$$\delta_t\big(O(T^{n-1}x;m-n+1)\big)=\eta\big(M(T^{n-1}x,T^{k+n-1}x,t)\big)\tag{2.6}$$

for all $t>0$. Because $n\geq 2$, it follows that $T^{n-2}x\in C_T$ and so putting $i=1$ and $j=k+1$ in Lemma 2.15, we obtain

$$\begin{aligned}\eta\big(M(T^{n-1}x,T^{k+n-1}x,t)\big)&=\eta\big(M(TT^{n-2}x,T^{k+1}T^{n-2}x,t)\big)\\&\leq\lambda\delta_t\big(O(T^{n-2}x;m-n+2)\big)\end{aligned}\tag{2.7}$$

for all $t>0$. Combining (2.5), (2.6) and (2.7) together with (2.4) and using induction, we get

$$\begin{aligned}\eta\big(M(T^mx,T^nx,t)\big)&\leq\lambda\delta_t\big(O(T^{n-1}x;m-n+1)\big)\\&=\lambda\eta\big(M(T^{n-1}x,T^{k+n-1}x,t)\big)\\&\leq\lambda^2\delta_t\big(O(T^{n-2}x;m-n+2)\big)\\&\ \ \vdots\\&\leq\lambda^n\delta_t\big(O(x;m)\big)\\&\leq\frac{\lambda^n}{1-\lambda}\limsup_{j\to\infty}\eta\big(M(x,Tx,\sum_{i=j+1}^{\infty}a_it)\big),\end{aligned}$$

which implies from (ii) that

$$\lim_{m,n\to\infty}\eta\big(M(T^mx,T^nx,t)\big)=0.$$

Hence, $\lim_{m,n\to\infty}M(T^nx,T^mx,t)=1$. This means that $\{T^nx\}$ is a Cauchy sequence. □

Following Petruşel and Rus [16], we introduce the concept of a Picard and weakly Picard operator in fuzzy metric spaces as follows.

Definition 2.20. Let $(X,M,\star)$ be a fuzzy metric space and $T:X\to X$ be a mapping.

(i) T is called a Picard operator if T has a unique fixed point $x^*\in X$ and $\lim_{n\to\infty}M(T^nx,x^*,t)=1$ for all $x\in X$ and each $t>0$.
(ii) T is called a weakly Picard operator if $\{T^nx\}$ is a convergent sequence and its limit (which depends on x) is a fixed point of T for all $x\in X$.

It is clear that each Picard operator is weakly Picard operator but the identity mapping of any fuzzy metric space with more than one point shows that the converse is not generally true. In fact, the set of fixed points of a weakly

Picard operator can have any arbitrary cardinality. Nevertheless, one can easily see that a weakly Picard operator is Picard one if and only if it has a unique fixed point.

Motivated by Jachymski [13], we define a weaker type of continuity of self-maps in fuzzy metric spaces endowed with a graph as follows.

Definition 2.21. Let $(X, M, \star)$ be a fuzzy metric space endowed with a graph G and $T : X \to X$ be a mapping. We say that T is orbitally G-continuous on X if $\lim_{n\to\infty} M\big(T^{a_n}x, y, t\big) = 1$ implies $\lim_{n\to\infty} M\big(T(T^{a_n}x), Ty, t\big) = 1$ for all $x, y \in X$ and each $t > 0$ and all sequences $\{a_n\}$ of positive integers such that $(T^{a_n}x, T^{a_n+1}x) \in E(G)$ for all $n \in \mathbb{N}$.

It is clear that a continuous mapping on a fuzzy metric space is orbitally G-continuous for all graphs G but the converse is not true in general as the following example shows.

Example 2.22. Let $(\mathbb{R}^+, d)$ be the usual (Euclidean) metric space of all non-negative real numbers and $(\mathbb{R}^+, M_d, \star)$ be the standard fuzzy metric space induced by d. Consider a mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by the rule

$$Tx = \begin{cases} \frac{x}{3}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then it is clear that T is not continuous at $x = 0$ and in particular, T is not continuous on the whole $\mathbb{R}^+$. Now, suppose that $\mathbb{R}^+$ is endowed with a graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}^+$ and $E(G) = \{(x, x) : x \in \mathbb{R}^+\}$, i.e. $E(G)$ contains all loops. If $x, y \in \mathbb{R}^+$ and $\{a_n\}$ is a sequence of positive integers with $(T^{a_n}x, T^{a_n+1}x) \in E(G)$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} M\big(T^{a_n}x, y, t\big) = 1$ for any $t > 0$, then $\{T^{a_n}x\}$ is necessarily a constant sequence. Hence, $T^{a_n}x = y$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} M\big(T(T^{a_n}x), Ty, t\big) = 1$ for each $t > 0$. Therefore, T is orbitally G-continuous on $\mathbb{R}^+$.

Now, we are ready to prove our main theorem on the existence of a fixed point for a G-fuzzy $\mathcal{H}$-quasi-contraction in the setup of M-complete fuzzy metric spaces endowed with a graph.

Theorem 2.23. *Let $(X, M, \star)$ be an M-complete fuzzy metric space endowed with a graph G and $T : X \to X$ be a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ such that*

(i) *$\tau \geq r \star s$ implies $\eta(\tau) \leq \eta(r) + \eta(s)$ for all $r, s, \tau \in \big\{M(T^i x, T^j x, t) : x \in X,\, t > 0,\, i, j \in \mathbb{N}\big\}$;*

(ii) *$\big\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\big\}$ is bounded for all $x \in X$ and any sequence $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.*

Then the restriction of T to C_T is a weakly Picard operator if either T is orbitally $\widetilde{G}$-continuous on X or G is a $(\widetilde{\mathrm{C}})$-graph.

In particular, whenever either T is orbitally $\widetilde{G}$-continuous on X or G is a $(\widetilde{\mathrm{C}})$-graph, T has a fixed point in X if and only if $C_T \neq \varnothing$.

Proof. If $C_T = \varnothing$, then there remains nothing to prove. So assume that C_T is nonempty. Note that since T preserves the edges of $\widetilde{G}$, it follows immediately that C_T is T-invariant, i.e. T maps C_T into itself.

Now, suppose that $x \in C_T$ is given. By virtue of Lemma 2.19, $\{T^n x\}$ is a Cauchy sequence. As $(X, M, \star)$ is M-complete, there exists $x^* \in X$ (depending on x) such that $\lim_{n\to\infty} M(T^n x, x^*, t) = 1$ for all $t > 0$. We shall show that x^* is a fixed point for T.

To this end, note that from $x \in C_T$, we have $(T^n x, T^{n+1}x) \in E(\widetilde{G})$ for all $n \in \mathbb{N} \cup \{0\}$. On the one hand, if T is orbitally $\widetilde{G}$-continuous on X, then $\lim_{n\to\infty} M(T^n x, x^*, t) = 1$ for all $t > 0$ implies $\lim_{n\to\infty} M(T^{n+1}x, Tx^*, t) = \lim_{n\to\infty} M(T(T^n x), Tx^*, t) = 1$ for all $t > 0$. By the uniqueness of the limit, we get $M(x^*, Tx^*, t) = 1$ for all $t > 0$, i.e. $Tx^* = x^*$.

On the other hand, if G is a $(\widetilde{\text{C}})$-graph, then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $(T^{n_k}x, x^*) \in E(\widetilde{G})$ for all $k \in \mathbb{N}$. Due to Proposition 2.14, if $\lambda \in (0, 1)$ is a quasi-contractive constant of T, then T is a $\widetilde{G}$-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ with a quasi-contractive constant λ. Let $t > 0$ is given. Hence, for all $\epsilon > 0$ and $k \in \mathbb{N}$, we have

$$M(Tx^*, x^*, t+\epsilon) \geq M(Tx^*, T^{n_k+1}x, \epsilon) \star M(T^{n_k+1}x, x^*, t)$$

which together with (i) and (FQ2) yields

$$\begin{aligned}\eta\big(M(x^*, Tx^*, t+\epsilon)\big) &\leq \eta\big(M(x^*, T^{n_k+1}x, \epsilon)\big) + \eta\big(M(Tx^*, T^{n_k+1}x, t)\big)\\ &\leq \eta\big(M(Tx^*, T^{n_k+1}x, t)\big) + \lambda \max\big\{\eta\big(M(x^*, T^{n_k}x, t)\big),\\ &\quad \eta\big(M(x^*, Tx^*, t)\big), \eta\big(M(T^{n_k}x^*, T^{n_k+1}x^*, t)\big),\\ &\quad \eta\big(M(x^*, T^{n_k+1}x^*, t)\big), \eta\big(M(T^{n_k}x, Tx^*, t)\big)\big\}.\end{aligned}$$

On taking the limit as $k \to \infty$ in the above inequality, we obtain

$$\eta\big(M(x^*, Tx^*, t+\epsilon)\big) \leq \lambda\eta\big(M(x^*, Tx^*, t)\big),$$

which implies that

$$\eta\big(M(x^*, Tx^*, t)\big) = \lim_{\epsilon\to 0^+} \eta\big(M(x^*, Tx^*, t+\epsilon)\big) \leq \lambda\eta\big(M(x^*, Tx^*, t)\big).$$

As $\lambda \in (0, 1)$, it then follows that $\eta\big(M(x^*, Tx^*, t)\big) = 0$ for all $t > 0$. Thus $M(x^*, Tx^*, t) = 1$ for all $t > 0$ or equivalently, $Tx^* = x^*$.

Finally, since C_T contains all fixed points of T, it follows that $x^* \in C_T$. Consequently, the restriction of $T\big|_{C_T} : C_T \to C_T$ is a weakly Picard operator. □

By putting $G = G_0$ in Theorem 2.23, we obtain the following generalization of Ćirić's fixed point theorem [7] on M-complete fuzzy metric spaces in the sense of George and Veeramani.

Corollary 2.24. *Let $(X, M, \star)$ be an M-complete fuzzy metric space and $T : X \to X$ be a fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ such that*

(i) *$\tau \geq r \star s$ implies $\eta(\tau) \leq \eta(r) + \eta(s)$ for all $r, s, \tau \in \big\{M(f^i x, f^j x, t) : x \in X,\, t > 0,\, i, j \in \mathbb{N}\big\}$;*

(ii) *$\big\{\eta(M(x, fx, t_i)) : i \in \mathbb{N}\big\}$ is bounded for all $x \in X$ and any sequence $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.*

Then T is a Picard operator.

Proof. The set C_T is nonempty because $C_T = X$. Therefore, due to Theorem 2.23, the mapping $T = T\big|_{C_T}$ is a weakly Picard operator. In particular, T has a fixed point in X. To see that T is a Picard operator, it sufficies to show that T has a unique fixed point in X. To this end, suppose that x^* and y^* are two fixed points for T in X. Thus, from (2.1), we have

$$\begin{aligned}\eta\big(M(x^*, y^*, t)\big) &= \eta\big(M(Tx^*, Ty^*, t)\big)\\ &\leq \lambda \max\big\{\eta\big(M(x^*, y^*, t)\big), \eta\big(M(x^*, Tx^*, t)\big), \eta\big(M(y^*, Ty^*, t)\big),\\ &\quad \eta\big(M(x^*, Ty^*, t)\big), \eta\big(M(y^*, Tx^*, t)\big)\big\}\\ &= \lambda\eta\big(M(x^*, y^*, t)\big),\end{aligned}$$

for all $t > 0$, where $\lambda \in (0, 1)$ is the quasi-contractive constant. Therefore, $\eta\big(M(x^*, y^*, t)\big) = 0$ which by our assumptions about $\eta \in \mathcal{H}$ implies that $M(x^*, y^*, t) = 1$ for all $t > 0$. Hence, $x^* = y^*$. □

Remark 2.25. By a subtle look at the proof of Corollary 2.24 and use an argument similar to that appeared there, we see that both the ends of any link of G can not be fixed points for a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$, i.e. if $x \neq y$, $Tx = x$ and $Ty = y$, then $(x, y) \notin E(G)$. Roughly speaking, no G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ can keep both the ends of a link of G fixed. In particular,

- if $G = G_0$, then no fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ can have two distinct fixed points;
- if $\preceq$ is a partial order on X, then neither a G_1-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ nor a G_2-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$ can have two distinct fixed points which are comparable elements of $(X, \preceq)$.

By taking $G = G_1$ or $G = G_2$ in Theorem 2.23, we obtain the ordered version of Ćirić's fixed point theorem on ordered fuzzy $\mathcal{H}$-quasi-contractions with respect to $\eta \in \mathcal{H}$ in M-complete fuzzy metric spaces equipped with a partial order as follows.

Corollary 2.26. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, \star)$ be an M-complete fuzzy metric space. Suppose that $T : X \to X$ be a mapping which maps comparable elements of $(X, \preceq)$ onto comparable elements and satisfies (2.2) such that for $\eta \in \mathcal{H}$,*

(i) $\tau \geq r \star s$ *implies* $\eta(\tau) \leq \eta(r) + \eta(s)$ *for all* $r, s, \tau \in \{M(f^i x, f^j x, t) : x \in X, t > 0, i, j \in \mathbb{N}\}$;

(ii) $\{\eta(M(x, fx, t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.

Then the restriction of T *to the set of all points* $x \in X$ *whose every two iterates under* T *are comparable elements of* $(X, \preceq)$ *is a weakly Picard operator if either* T *is orbitally* G_2*-continuous on* X *or the quadruple* $(X, M, \star, \preceq)$ *satisfies* $(*)$.

In particular, whenever either T *is orbitally* G_2*-continuous on* X *or the quadruple* $(X, M, \star, \preceq)$ *satisfies* $(*)$, T *has a fixed point in* X *if and only if there exists* $x \in X$ *such that* $T^m x$ *and* $T^n x$ *are comparable elements of* $(X, \preceq)$ *for all* $m, n \in \mathbb{N} \cup \{0\}$.

Because G-fuzzy $\mathcal{H}$-Banach contractions with respect to $\eta \in \mathcal{H}$, G-fuzzy $\mathcal{H}$-Kannan contractions with respect to $\eta \in \mathcal{H}$, G-fuzzy $\mathcal{H}$-Chatterjea contractions with respect to $\eta \in \mathcal{H}$, G-fuzzy $\mathcal{H}$-Ćirić-Reich-Rus contractions with respect to $\eta \in \mathcal{H}$ and G-fuzzy $\mathcal{H}$-Hardy-Rogers contractions with respect to $\eta \in \mathcal{H}$ are all a G-fuzzy $\mathcal{H}$-quasi-contraction with respect to $\eta \in \mathcal{H}$, we have also the following fixed point theorem for these contractions as a consequence of Theorem 2.23.

Corollary 2.27. *Let* $(X, M, \star)$ *be an* M*-complete fuzzy metric space endowed with a graph* G *and* $T : X \to X$ *be a* G*-fuzzy* $\mathcal{H}$*-Banach contraction (a* G*-fuzzy* $\mathcal{H}$*-Kannan contraction, a* G*-fuzzy* $\mathcal{H}$*-Chatterjea contraction, a* G*-fuzzy* $\mathcal{H}$*-Ćirić-Reich-Rus contraction, or a* G*-fuzzy* $\mathcal{H}$*-Hardy-Rogers contraction) with respect to* $\eta \in \mathcal{H}$ *such that*

(i) $\tau \geq r \star s$ *implies* $\eta(\tau) \leq \eta(r) + \eta(s)$ *for all* $r, s, \tau \in \{M(f^i x, f^j x, t) : x \in X, t > 0, i, j \in \mathbb{N}\}$;

(ii) $\{\eta(M(x, fx, t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_n\} \subseteq (0, \infty)$, $t_n \downarrow 0$.

Then the restriction of T *to* C_T *is a weakly Picard operator if either* T *is orbitally* $\widetilde{G}$*-continuous on* X *or* G *is a* $(\widetilde{\mathrm{C}})$*-graph.*

In particular, whenever either T *is orbitally* $\widetilde{G}$*-continuous on* X *or* G *is a* $(\widetilde{\mathrm{C}})$*-graph,* T *has a fixed point in* X *if and only if* $C_T \neq \varnothing$.

Remark 2.28. By comparing Corollary 2.27 as a version of Theorem 2.23 for several types of contractions with some recent results in graph metric fixed point theory, one can see easily that our results can be viewed as the improvement and generalization of corresponding results in [3, 4, 6, 12, 13, 14, 18, 17] and several other comparable results.

3. Conclusion

Despite noted improvements in computer skill and its remarkable success in facilitating many areas of research, computers are not designed to handle situations wherein uncertainties are involved. Fuzzy set theory has provided many important tools in mathematics and related disciplines to resolve the issues of uncertainty and ambiguity. In the present work, we investigated sufficient conditions which guarantee the existence of a fixed point for a new notion called G-fuzzy $\mathcal{H}$-quasi-contraction using directed graphs in the setting of fuzzy metric spaces endowed with a graph. A large number of different types of contractive mappings formulated using directed graphs satisfy the presented contractive condition and our main result is a natural generalization of [2, Definition 2.3] from fuzzy metric spaces to fuzzy metric spaces with a graph and enriches our knowledge of fixed points in such spaces. As a new work, it will be interesting to study common fixed point results for two or more than two mappings on fuzzy metric spaces endowed with a graph G by considering the function $\eta \in \mathcal{H}$.

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